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On q-functional equations and excursion moments

Christoph Richard

Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, 33501 Bielefeld, Germany

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Abstract

We analyse q-functional equations arising from tree-like combinatorial structures, which are counted by size, internal path length, and certain generalisations thereof. The corresponding counting parameters are labelled by a positive integer k. We show the existence of a joint limit distribution for these parameters in the limit of infinite size, if the size generating function has a square root as dominant singularity. The limit distribution coincides with that of integrals of kth powers of the standard Brownian excursion. Our approach yields a recursion for the moments of the limit distribution. It can be used to analyse asymptotic expansions of the moments, and it admits an extension to other types of singularity.

Keywords: Simply generated trees; q-difference equation; Brownian excursion; Limit distribution

1. Introduction and main results

1.1. A combinatorial motivation

When studying combinatorial classes, a functional equation of the form

$$G(x) = P(x, G(x)) \tag{1.1}$$

frequently arises, where $G(x)$ is the generating function of the class, and $P(x, y)$ is a formal power series in two variables with real coefficients. Prominent examples are classes of simply generated trees, counted by number of vertices [37], classes of directed square lattice paths counted by length [4], and classes of square lattice polygons, counted by perimeter [9]. Indeed, there exist combinatorial bijections between corresponding models of trees, paths, and polygons, see e.g. [49] for Catalan trees, Dyck paths, and staircase polygons. Eq. (1.1) reflects a combinatorial decomposition of the given class. If $P(0, y) \equiv 0$ and $\frac{\partial P}{\partial y}(0, y) \not\equiv 1$, then Eq. (1.1) admits a unique solution $G(x) \in \mathbb{R}[[x]]$ such that $G(0) = 0$. Here, $\mathbb{R}[[x]]$ denotes the ring of formal power series in x with coefficients in $\mathbb{R}$. Often, the coefficients of $P(x, y)$ are non-negative real numbers. Then, the series $G(x)$ is a power series with non-negative coefficients, typically analytic at $x = 0$ with a square root as dominant singularity, see e.g. [41, Thm. 10.6], [16], or [24, Ch. VII.4], and references therein.

E-mail address: richard@math.uni-bielefeld.de.

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We are interested in certain deformations of the above equation. This is done by introducing a new variable q, such that the limit $q \to 1$ reduces to the original equation. For example, the functional equation

$$G(x, q) = P(x, G(qx, q)) \tag{1.2}$$

defines a formal power series in x with coefficients polynomially in q, i.e., $G(x, q) \in \mathbb{R}[q][[x]]$, if Eq. (1.1) defines a power series $G(x) \in \mathbb{R}[[x]]$. The above equation is a *q-difference equation*, see e.g. [51] and references therein. It appears in classes of simply generated trees, counted by number of vertices and internal path length (i.e., the sum of the vertex distances to the root), in classes of directed square lattice paths, counted by length and area under the path, and in classes of square lattice polygons, counted by perimeter and area. For some models, an explicit expression for its generating function $G(x, q)$ is known, see e.g. [9,44,45] and references therein. Such an expression typically contains q-products and has a natural boundary $|q| = 1$. An interesting question concerns the statistics of the additional counting parameter, e.g., in a uniform ensemble in the limit of infinite system size. It is known ([50, 19], see also [46,43]) that Eq. (1.2) leads, for certain q-difference equations and after appropriate normalisation, to the *Airy distribution* as the limit distribution for the additional parameter. This distribution is known to also describe the area under a Brownian excursion, see the following subsection. Note that generalisations of Eq. (1.1) other than Eq. (1.2) have also been studied previously. A class of equations, which lead to Gaussian limit laws, is discussed in [16].

The idea of iterating the above deformations has been considered by Duchon [18,19]. The deformation variables may be denoted by q_k, where $k \in \{1, \dots, M\}$, and the associated counting parameters are called *parameters of rank* $k + 1$. For Eq. (1.2), an example is given by

$$G(x, q_1, \dots, q_M) = P(x, G(xq_1 \cdot \dots \cdot q_M, q_1 q_2 \cdot \dots \cdot q_M, q_2 q_3 \cdot \dots \cdot q_M, \dots, q_M)). \tag{1.3}$$

Here, $G(x, q_1, \dots, q_M) = \sum_n p_n(q_1, \dots, q_M)x^n$ is a formal power series in x, with polynomial coefficients $p_n(q_1, \dots, q_M) \in \mathbb{R}[q_1, \dots, q_M]$. The name "parameter of rank $k + 1$" reflects that k is the smallest integer r, such that the degree of the polynomial $p_n(q_1, \dots, q_M)$ in q_k is bounded by $c\, n^{r+1}$ for some constant c (see [19, Lemma 1] and [18]). We will call an equation like Eq. (1.3) a *q-functional equation* (Definition 2.4). Again the question arises, under which conditions a limit distribution for the additional counting parameters exists. In this paper, we shall show this for a class of deformations which we call *q-shift* (Definition 2.1), the main assumption on the q-functional equation being that the solution of the undeformed equation, Eq. (1.1), is analytic at the origin, with a square root as dominant singularity (Assumption 4.1). See Theorem 1.5 for a precise statement. The resulting limit distributions appear to be related to distributions of integrals of kth powers of the standard Brownian excursion. We will obtain a recursion for the moments of the joint distribution. Our approach is based on the multivariate *moment method* (see e.g. [27,7]). The univariate case $M = 1$ has been studied previously [50,23,19,46], and recursions for $M = 2$ have been studied in [38,40].

Before we consider general q-functional equations in Section 2, let us first discuss the above questions in more detail for the simple example of Dyck paths.

1.2. Dyck paths and Brownian excursions

We review the connection between Dyck paths and Brownian excursions. This relates, in particular, the moments of height of a random Dyck path to Brownian excursion moments. Since the generating function of Dyck paths, counted by length and moments of their height, provides a simple example of a solution of a q-functional equation, this also motivates the appearance of excursion moments in q-functional equations. We will state a central result of the paper in Theorem 1.2, which characterises the Brownian excursion random variables by a recursion for their moments.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. A *Dyck path of length* $2n$, where $n \in \mathbb{N}_0$, is a map $y : [0, 2n] \to \mathbb{R}_{\geq 0}$, where $y(0) = y(2n) = 0$ and $|y(i) - y(i + 1)| = 1$ for $i \in \mathbb{N}_0$ and $i < 2n$. For non-integral argument, $y(s)$ is defined by linear interpolation. The values $y(s)$ are called the *heights* of the path. An *arch of length* $2n$ is a Dyck path of length $2n$, where $n > 0$ and $y(s) > 0$ for $s \in (0, 2n)$. An example is given in Fig. 1.

We are interested in a probabilistic interpretation of Dyck paths in a uniform ensemble where, for fixed length, each of the finitely many Dyck paths occurs with the same probability. For $0 \leq s \leq 2n$, let $\widetilde{Y}_n(s)$ denote the height of a random Dyck path of length $2n$. It is well known, see e.g. [13], that the average maximal height $\widetilde{h}(n)$ of a random Dyck path of length $2n$ has the asymptotic form $\widetilde{h}(n) \sim \sqrt{\pi n}$ as $n \to \infty$. In order to obtain a finite

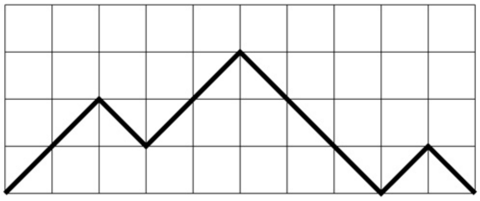

Fig. 1. A Dyck path of length $2n = 10$. It is a sequence of two arches of lengths 8 and 2.

positive limit as n approaches infinity, and in order to normalise the domain, we introduce the normalised height $Y_n(t) = (2n)^{-1/2}\widetilde{Y}_n(2nt)$, where $0 \le t \le 1$.

The sequence $\{Y_n(t)\}_{n\in\mathbb{N}}$ is a sequence of stochastic processes defined on $(C[0, 1], \|\cdot\|_\infty)$ with the Borel σ-algebra, which converges in distribution to the standard Brownian excursion $e(t)$ of duration 1, see [6,1–3,36]. This implies convergence in distribution of sequences of continuous bounded functionals of $Y_n(t)$ towards the corresponding excursion functionals. (We refer to [8] for background about convergence of probability measures.) There is a more recent result [17, Thm. 9] that also

$$\lim_{n\to\infty} \mathbb{E}_n[F(Y_n(t))] = \mathbb{E}[F(e(t))]$$

for continuous functionals $F : C[0, 1] \to \mathbb{R}$ of *polynomial* growth, i.e., for functionals such that there exists an $r \ge 0$ with $|F(y)| \le \|y\|_\infty^r$ for all $y \in C[0, 1]$. The above property is called *polynomial convergence*. In particular, polynomial convergence implies *moment convergence* for such a functional, i.e., convergence of the sequence of moments of order l, $\{\mathbb{E}_n[F(Y_n(t))^l]\}_{n\in\mathbb{N}}$, for every $l \in \mathbb{N}_0$. For the functional of polynomial growth $F(y) = \int_0^1 y^k(t)\mathrm{d}t$, $k \in \mathbb{N}$, we call the random variable $\int_0^1 e^k(t)\mathrm{d}t$ the kth *excursion moment*.

As counting parameters for Dyck paths of length $2n$, we consider the parameters

$$x_{k,n} = \sum_{i=0}^{2n} y^k(i) \quad (k = 1, \ldots, M). \tag{1.4}$$

These are sums of kth powers of heights, which we call kth *moments of height*. The parameter $x_{1,n}$ is the area under the Dyck path, the parameter $x_{2,n}$ is called the *moment of inertia* of the Dyck path [40]. Let $\widetilde{X}_{k,n}$ denote the kth moment of height of a random Dyck path of length $2n$. In terms of $\widetilde{Y}_n(s)$, the random variables $\widetilde{X}_{k,n}$ are expressed as

$$\widetilde{X}_{k,n} = \sum_{i=0}^{2n} \widetilde{Y}_n^k(i).$$

The scaling of the average height of a random Dyck path with its length suggests considering normalised random variables $X_{k,n}$, defined by

$$(X_{1,n}, X_{2,n}, \ldots, X_{M,n}) = \left(\frac{\widetilde{X}_{1,n}}{n^{(1+2)/2}}, \frac{\widetilde{X}_{2,n}}{n^{(2+2)/2}}, \ldots, \frac{\widetilde{X}_{M,n}}{n^{(M+2)/2}}\right). \tag{1.5}$$

Then, in terms of $Y_n(t)$, the normalised random variables $X_{k,n}$ are given by

$$X_{k,n} = 2^{(k+2)/2} \sum_{i=0}^{2n} Y_n^k\left(\frac{i}{2n}\right) \cdot \frac{1}{2n}. \tag{1.6}$$

Since the above sum resembles a Riemann sum, one is led to expect convergence in distribution and moment convergence of the sequence $\{X_{k,n}\}_{n\in\mathbb{N}}$ to $2^{(k+2)/2}\int_0^1 e^k(t)\mathrm{d}t$, due to the convergence properties of $\{Y_n(t)\}_{n\in\mathbb{N}}$. This is the statement of the following proposition.

Proposition 1.1. *For $k \in \{1, \ldots, M\}$, let $X_k = \int_0^1 e^k(t)\,\mathrm{d}t$ denote the kth excursion moment. The sequence of Dyck path random variables Eq.* (1.5) *converges to the normalised excursion moments* $(2^{(1+2)/2}X_1, \ldots, 2^{(M+2)/2}X_M)$ *in* *distribution,*

$$(X_{1,n}, X_{2,n}, \ldots, X_{M,n}) \xrightarrow{d} (2^{(1+2)/2}X_1, 2^{(2+2)/2}X_2, \ldots, 2^{(M+2)/2}X_M) \quad (n \to \infty). \tag{1.7}$$

We also have moment convergence.

Proof. For $k \in \{1, \ldots, M\}$, define $Y_{k,n} = \int_0^1 Y_n^k(t)\mathrm{d}t$. As stated above, convergence in distribution and moment convergence holds for the sequence $\left\{(Y_{1,n}, \ldots, Y_{M,n})\right\}_{n\in\mathbb{N}}$, see [6,3] and [17, Thm. 9]. We argue that the sequences $\{(X_{1,n}, \ldots, X_{M,n})\}_{n\in\mathbb{N}}$ and $\{(2^{(1+2)/2}Y_{1,n}, \ldots, 2^{(M+2)/2}Y_{M,n})\}_{n\in\mathbb{N}}$ converge in distribution and for moments to the same limit.

Fix $k \in \{1, \ldots, M\}$ and consider, for a Dyck path y of length $2n$ and for $m \in \{0, 1, \ldots, 2n-1\}$, the elementary estimate $(\max\{0, y(m)-1\})^k \le \int_m^{m+1} y^k(s)\mathrm{d}s \le (y(m)+1)^k$. Summing over m yields, together with the binomial theorem, the estimate

$$\sum_{l=0}^{k} \binom{k}{l} (-1)^l x_{l,n} \le \int_0^{2n} y^k(s)\mathrm{d}s \le \sum_{l=0}^{k} \binom{k}{l} x_{l,n}. \tag{1.8}$$

This estimate translates to the distribution functions of the corresponding random variables. In terms of the random variables $X_{k,n}$ and $Y_n(t)$, we get

$$\sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{X_{l,n}}{n^{(k-l)/2}} \overset{d}{\le} 2^{(k+2)/2} \int_0^1 Y_n^k(t)\mathrm{d}t \overset{d}{\le} \sum_{l=0}^{k} \binom{k}{l} \frac{X_{l,n}}{n^{(k-l)/2}}, \tag{1.9}$$

where the superscript ^{d} indicates that the relation is to be understood via distribution functions. Since the sequence of random variables $\{X_{l,n}/n^{(l-k)/2}\}_{n\in\mathbb{N}}$ converges in probability to zero, for $l = 0, \ldots, k-1$, the random variables on the l.h.s. and on the r.h.s. of Eq. (1.9) converge in distribution to the same limit. We conclude that the sequences $\{X_{k,n}\}_{n\in\mathbb{N}}$ and $\{2^{(k+2)/2}\int_0^1 Y_n^k(t)\mathrm{d}t\}_{n\in\mathbb{N}}$ converge in distribution to the same limit. Moment convergence follows similarly from Eq. (1.8).

Since the above estimates also hold jointly in $k \in \{1, \ldots, M\}$, the statement of the proposition follows. $\square$

Remark. The above statement concerns Dyck paths in a uniform ensemble. The same argument as above leads to analogous results for more general ensembles of Dyck paths, which arise from the depth first process of simply generated trees [1–3,36]. Compare to Theorem 1.5 for a further generalisation.

Due to the above result, Brownian excursion functionals may be studied via their discrete Dyck path counterparts. Below, we will analyse the asymptotic behaviour of the moments of the random variables $(X_{1,n}, \ldots, X_{M,n})$ Eq. (1.5), using the q-functional equation which the generating function of Dyck paths obeys. This implies a certain recursion for the moments of the joint distribution of the excursion moments $(X_1, \ldots, X_M)$. We have the following result, which is obtained by combining the statements of Propositions 1.1 and 1.4. For M-dimensional vectors, we use the abbreviation $\boldsymbol{k} = (k_1, \ldots, k_M)$ and write $\boldsymbol{0} = (0, \ldots, 0)$. For $i \in \{1, \ldots, M\}$, the unit vector $\boldsymbol{e}_i$ is defined by $(\boldsymbol{e}_i)_j = \delta_{i,j}$ for $j = 1, \ldots, M$, and $\boldsymbol{l} \le \boldsymbol{k}$ for vectors $\boldsymbol{l} = (l_1, \ldots, l_M)$ and $\boldsymbol{k} = (k_1, \ldots, k_M)$ means that $l_i \le k_i$ for $i = 1, \ldots, M$. A multivariate power series with complex coefficients is called *entire*, if it converges for arbitrary complex arguments.

Theorem 1.2 (*Excursion Moments*). *For $k \in \{1, \ldots, M\}$, let $X_k = \int_0^1 e^k(t)\,\mathrm{d}t$ denote the kth excursion moment. The moments of the joint distribution of $(X_1, \ldots, X_M)$ are given by*

$$\frac{\mathbb{E}[X_1^{k_1} \cdot \ldots \cdot X_M^{k_M}]}{k_1! \cdot \ldots \cdot k_M!} = \frac{\sqrt{2\pi}}{\Gamma(\gamma_{k_1,\ldots,k_M})} 2^{\gamma_{k_1,\ldots,k_M}} \frac{f_{k_1,\ldots,k_M}}{2}, \tag{1.10}$$

where $\gamma_{k_1,\ldots,k_M} = \gamma_{\boldsymbol{k}} = -1/2 + \sum_{i=1}^{M}(1+i/2)k_i$, and where $\Gamma(z)$ is the Gamma function. For $\boldsymbol{k} \ne \boldsymbol{0}$, the numbers $f_{k_1,\ldots,k_M} = f_{\boldsymbol{k}}$ are characterised by the recursion

$$\gamma_{\boldsymbol{k}-\boldsymbol{e}_1} f_{\boldsymbol{k}-\boldsymbol{e}_1} + \sum_{i=1}^{M-1} 2(i+1)(k_i+1) f_{\boldsymbol{k}-\boldsymbol{e}_{i+1}+\boldsymbol{e}_i} + \sum_{\boldsymbol{0}\le\boldsymbol{l}\le\boldsymbol{k}} f_{\boldsymbol{l}} f_{\boldsymbol{k}-\boldsymbol{l}} = 0, \tag{1.11}$$

with boundary conditions $f_{\mathbf{0}} = -4$ *and* $f_{\mathbf{k}} = 0$*, if* $k_j < 0$ *for some* $j \in \{1, \ldots, M\}$*. The moment generating function* $\mathbb{E}[e^{t_1 X_1 + \ldots + t_M X_M}]$ *is entire. Hence, the joint distribution is uniquely defined by its moments.* □

Remark. (i) The theorem asserts that the numbers $f_{\mathbf{k}}$ can be recursively computed from Eq. (1.11). To see this, we remark that the above equation has $-f_{\mathbf{k}} = 2 f_{\mathbf{k}} f_{\mathbf{0}}/8$ as a term, and that all other numbers $f_{\mathbf{l}}$ in the equation satisfy $\mathbf{l} \prec \mathbf{k}$ for a suitable total order $\prec$, which is specified below in Definition 2.2.

(ii) In probability theory, Louchard's theorem [34] leads to a characterisation of a certain Laplace transform of the moment generating function, for some excursion functionals including excursion moments. It is however difficult to extract moment values or moment recursions from these expressions. For $M = 1$, this has been done in [35]. The moment generating function of the marginal distribution for $M = 2$ has been obtained in [40, Thm. 2.4]. Moment values or recursions for $M > 2$ have apparently not been derived previously.

(iii) We obtain Theorem 1.2 by studying the corresponding Dyck path functionals. Our discrete approach has been used previously [38,40]. It led to a combinatorial derivation [40, Thm. 3.1] of Louchard's formula for integrals over Brownian excursion polynomials. It also led to the values $\mathbb{E}[(X_1)^{k_1}(X_2)^{k_2}]$, see [40, Table 2]. For arbitrary M, our result was announced in [47]. [A factor 1/2 is missing in the r.h.s. of Eq. (6) in [47]].

(iv) Assuming only convergence in distribution of $(X_{1,n}, \ldots, X_{M,n})$, the above result can be used to provide an alternative proof of moment convergence.

Let us briefly discuss the moments of the marginal distributions for $M \le 4$. For $M = 1$, the first few coefficients $\mathbb{E}[(X_1)^k]$ are 1, $\frac{1}{4}\sqrt{2\pi}$, $\frac{5}{12}$, $\frac{15}{128}\sqrt{2\pi}$, $\frac{221}{1008}$, $\frac{565}{8192}\sqrt{2\pi}$. These are related to the *Airy distribution* [21,30,31], i.e., the distribution of area under the Brownian excursion $\sqrt{8}\, e(t)$. Explicit expressions are known for the moment generating function, for the density of the distribution, and for its moments.

For $M = 2$, the first few moments $\mathbb{E}[(X_2)^k]$ are 1, $\frac{1}{2}$, $\frac{19}{60}$, $\frac{631}{2520}$, $\frac{1219}{5040}$, $\frac{92\,723}{332\,640}$, $\frac{1513\,891}{4036\,032}$. An explicit expression for the corresponding moment generating function [40] is

$$\mathbb{E}[e^{tX_2}] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[(X_2)^k]}{k!} t^k = \left(\frac{\sqrt{2t}}{\sin(\sqrt{2t})} \right)^{3/2}.$$

The first few moments $\mathbb{E}[(X_3)^k]$ are 1, $\frac{3\sqrt{2\pi}}{16}$, $\frac{207}{560}$, $\frac{11\,907\sqrt{2\pi}}{65\,536}$, $\frac{88\,655\,283}{1089\,088\,00}$, $\frac{1165\,359\,069\sqrt{2\pi}}{1476\,395\,008}$, and for the first few moments $\mathbb{E}[(X_4)^k]$ we have 1, $\frac{1}{2}$, $\frac{251}{840}$, $\frac{288\,751}{1201\,200}$, $\frac{19\,093\,793}{76\,236\,160}$, $\frac{105\,169\,404\,203}{3259\,095\,840\,000}$. It remains an open problem to find explicit expressions for the moment generating functions.

1.3. Dyck paths and *q*-functional equations

We discuss the functional equation which the generating function of Dyck paths obeys, when counted by length and kth moments of height, where $k = 1, \ldots, M$. We then state in Theorem 1.5 a convergence result for the limit distribution of counting parameters related to general q-functional equations. This is the main result of our paper.

Let $\boldsymbol{u} = (u_0, u_1, \ldots, u_M)$ denote formal (commutative) variables. For a Dyck path p of length $2n$, its *weight* $w(p)$ is given by $w(p) = u_0^n u_1^{x_{1,n}} \cdot \cdots \cdot u_M^{x_{M,n}}$, where $x_{k,n}$ is the kth moment of height equation (1.4). For the set $\mathcal{D}$ of Dyck paths, its generating function is the formal power series $D(\boldsymbol{u}) = \sum_{p \in \mathcal{D}} w(p)$. For the set $\mathcal{A}$ of arches, its generating function is the formal power series $A(\boldsymbol{u}) = \sum_{p \in \mathcal{A}} w(p)$. The variables $u_1, \ldots, u_M$ are interpreted as deformation variables. They may all be set equal to unity, in which case the generating function reduces to that of Dyck paths (arches), counted by length only. The generating functions satisfy the following functional equations.

Theorem 1.3 (*Dyck Path Generating Function, cf. [40]*). *The generating functions of Dyck paths* $D(\boldsymbol{u})$ *and of arches* $A(\boldsymbol{u})$ *satisfy*

$$\begin{aligned} D(u_0, \ldots, u_M) &= 1 + D(u_0, \ldots, u_M) A(u_0, \ldots, u_M), \\ A(u_0, u_1, \ldots, u_M) &= u_0 u_1 \cdot \ldots \cdot u_M D(v_0, \ldots, v_M), \end{aligned} \tag{1.12}$$

where the monomials $v_k = v_k(\boldsymbol{u})$ *are given by*

$$v_0(\boldsymbol{u}) = u_0 u_1^2 u_2^2 \cdot \ldots \cdot u_M^2, \qquad v_k(\boldsymbol{u}) = \prod_{l=k}^{M} u_l^{\binom{l}{k}} \quad (k = 1, \ldots, M). \quad \square$$

Remark. (i) For $M = 2$, the above theorem appears in [40, Sec. 3.1]. For arbitrary M, it is (incorrectly) stated in [40, p. 713]. The proof for $M = 2$ given in [40] generalises to arbitrary M. It uses a last passage decomposition argument and the additivity of the counting parameters with respect to the sequence construction, yielding the first equation in the theorem. Note that the arch decomposition of Dyck paths, as depicted in Fig. 1, implies the equivalent equation $D(\boldsymbol{u}) = 1/(1 - A(\boldsymbol{u}))$. For $n \in \mathbb{N}$, there is a bijection between the set of arches of length $2n$ and the set of Dyck paths of length $2n - 2$, by identifying an arch, where the bottom layer has been removed, with the corresponding Dyck path. This implies, after a short calculation (see also [48, Thm. 1]), the second equation in Theorem 1.3.

(ii) The induced functional equation for the generating function $E(\boldsymbol{u}) = D(\boldsymbol{u}) - 1$ is an example of a q-functional equation as in Definition 2.4, with a square-root singularity in the generating function in the "undeformed" case $(u_1, \dots, u_M) = (1, \dots, 1)$. See the following section and Section 8.

For a probabilistic interpretation of the counting parameters, as in the previous subsection, let $p_{n_0,n_1,\dots,n_M}$ denote the number of Dyck paths of length $2n_0$, where the kth moment of height has the value n_k, for $k = 1, \dots, M$. We clearly have $0 < \sum_{n_1,\dots,n_M} p_{n_0,n_1,\dots,n_M} < \infty$ for $n_0 > 0$. In the uniform ensemble, the random variables $\widetilde{X}_{k,n_0}$, which assign the kth moment of height to a random Dyck path of length $2n_0$, have the joint distribution

$$\mathbb{P}(\widetilde{X}_{1,n_0} = n_1, \dots, \widetilde{X}_{M,n_0} = n_M) = \frac{p_{n_0,n_1,\dots,n_M}}{\sum\limits_{n_1,\dots,n_M} p_{n_0,n_1,\dots,n_M}}. \tag{1.13}$$

The moments $\widetilde{m}_{k_1,\dots,k_M}(n_0)$ of the joint distribution are given by

$$\widetilde{m}_{k_1,\dots,k_M}(n_0) = \frac{\sum\limits_{n_1,\dots,n_M} n_1^{k_1} \cdot \dots \cdot n_M^{k_M} p_{n_0,n_1,\dots,n_M}}{\sum\limits_{n_1,\dots,n_M} p_{n_0,n_1,\dots,n_M}}.$$

In the previous subsection, we introduced the normalised random variable Eq. (1.5),

$$(X_{1,n_0}, X_{2,n_0}, \dots, X_{M,n_0}) = \left(\frac{\widetilde{X}_{1,n_0}}{n_0^{(1+2)/2}}, \frac{\widetilde{X}_{2,n_0}}{n_0^{(2+2)/2}}, \dots, \frac{\widetilde{X}_{M,n_0}}{n_0^{(M+2)/2}} \right). \tag{1.14}$$

The normalised random variable $(X_{1,n_0}, \dots, X_{M,n_0})$ has moments $m_{k_1,\dots,k_M}(n_0)$, given by

$$m_{k_1,\dots,k_M}(n_0) = \frac{\widetilde{m}_{k_1,\dots,k_M}(n_0)}{n_0^{(1+2)k_1/2+(2+2)k_2/2+\dots+(M+2)k_M/2}}.$$

We argued above that these numbers should tend to a finite limit, as n_0 approaches infinity. As we will see below, this is indeed the case. Hence we may define

$$m_{k_1,\dots,k_M} = \lim_{n_0\to\infty} m_{k_1,\dots,k_M}(n_0). \tag{1.15}$$

A careful analysis of the functional equation (1.12), which will be performed in the general case from Section 4 onwards, yields the following result for the numbers $m_{k_1,\dots,k_M}$. Its proof is deferred until Section 8.

Proposition 1.4. *The normalised moments $m_{k_1,\dots,k_M}$ of Dyck paths Eq. (1.15) are given by*

$$\frac{m_{k_1,\dots,k_M}}{k_1! \cdot \dots \cdot k_M!} = \frac{1}{f_{\boldsymbol{0}} u_c^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}}} \frac{\Gamma(\gamma_{\boldsymbol{0}})}{\Gamma(\gamma_{\boldsymbol{k}})} f_{\boldsymbol{k}}, \tag{1.16}$$

where $u_c = 1/4$, where $\Gamma(z)$ denotes the Gamma function, and where the numbers $\gamma_{\boldsymbol{k}}$ and $f_{\boldsymbol{k}}$ are defined in Eq. (1.11). The moments have an entire exponential generating function. Hence, they define a unique random variable with moments $m_{\boldsymbol{k}}$.

Remark. (i) Proposition 1.4 implies convergence in distribution and moment convergence of the normalised random variables Eq. (1.14). This yields an alternative proof of the convergence statement of Proposition 1.1. However, it does not establish a connection between the limit random variable and the excursion moments, as in Proposition 1.1.

(ii) Since the moments m_k define a unique random variable, the result for the excursion moments Eq. (1.10) follows from Proposition 1.4 by Proposition 1.1.

For the solution of a general q-functional equation (Definition 2.4), we obtain a similar result. Write $G(\boldsymbol{u}) = \sum_{n_1,\dots,n_M} p_{n_0,n_1,\dots,n_M} u_0^{n_0} u_1^{n_1} \cdot \ldots \cdot u_M^{n_M}$ for such a solution. Under mild assumptions, the corresponding random variables in Eq. (1.13) are well defined, and the same asymptotic analysis as above can be performed. We have the following theorem which is, in conjunction with Theorem 1.2, the main result of our paper. Its proof is deferred until Sections 7 and 8.

Theorem 1.5 (*Limit Distribution for q-Functional Equations*). *Let the power series*

$$G(u_0, u_1, \dots, u_M) = \sum_{n_1,\dots,n_M} p_{n_0,n_1,\dots,n_M} u_0^{n_0} u_1^{n_1} \cdot \ldots \cdot u_M^{n_M}$$

be the solution of a q-functional equation (see Definition 2.4) that is unique by Proposition 2.5, and let Assumption 4.1 be satisfied. Assume that for $i = 0, \dots, M-1$ the numbers A_i in Proposition 4.5 are positive. Then, the random variables $(\widetilde{X}_{1,n_0}, \dots, \widetilde{X}_{M,n_0})$ Eq. (1.13) are well defined for almost all n_0. The following conclusions hold.

(i) *The numbers $m_{k_1,\dots,k_M}$ Eq. (1.15), which are derived from the moments of the normalised random variables $(X_{1,n_0}, \dots, X_{M,n_0})$ Eq. (1.14), define a unique random variable $(Y_1, \dots, Y_M)$ with moments $m_{k_1,\dots,k_M}$. We have convergence in distribution,*

$$(X_{1,n_0}, \dots, X_{M,n_0}) \xrightarrow{d} (Y_1, \dots, Y_M) \quad (n_0 \to \infty),$$

and we have moment convergence.

(ii) *The limiting random variable $(Y_1, \dots, Y_M)$ is explicitly given by*

$$(Y_1, \dots, Y_k) = (c_1 X_1, \dots, c_M X_M),$$

with equality in the sense of distribution, where the constants $c_k > 0$ and the random variables X_k are

$$c_k = 2^{\frac{k+2}{2}} \frac{2\mu_0 \cdot \ldots \cdot \mu_{k-1}}{4^k k!}, \qquad X_k = \int_0^1 e^k(t)\,\mathrm{d}t \quad (k = 1, \dots, M).$$

Here, the numbers $\mu_i > 0$ are defined in Proposition 4.5, and $e(t)$ denotes a standard Brownian excursion of duration 1.

Remark. (i) A recursion for the moments of the limit distribution appears in Theorem 1.2, see also Proposition 4.5.

(ii) For Dyck paths counted by length and kth moments of height, the above statements follow already from Proposition 1.1. More generally, as was argued in the remark following Proposition 1.1, the above result can be shown to hold for models of simply generated trees [37], counted by number of vertices and kth moments of internal path length. This follows from the polynomial convergence of the depth first process derived from simply generated trees towards the Brownian excursion [6,1–3,36,17]. For a general q-functional equation, such a connection is not known to exist.

(iii) Our method of proof also allows one to study corrections to the asymptotic behaviour, compare the discussion in Section 5, and [46] for the case $M = 1$. These cannot be obtained by the methods of [36].

1.4. Structure of the paper

The remainder of the paper is organised as follows. In Section 2, we introduce q-shifts (Definition 2.1) and q-functional equations (Definition 2.4). Our results rely on an application of the multivariate moment method (see e.g. [27, Ch. 6.1.] or [7, Sec. 30]). Hence in Section 3, Eq. (2.9), we introduce *factorial moment generating functions*, which are derivatives of the solution of the functional equation, evaluated at $u_1 = \cdots = u_M = 1$. We study their properties by a combinatorial analysis of derivatives of the functional equation, using a multivariate generalisation of Faa di Bruno's formula [15]. In Section 4, we study the singular behaviour of the factorial moment generating functions, in the case of a square-root singularity as the dominant singularity of the size generating function. Proposition 4.5 gives a recursion for the amplitudes, which describe the leading singular behaviour of the factorial moment generating functions. Our method is also called *moment pumping* [23]. We employ in Section 5 an alternative (rigorous) method to obtain the recursion, which originates from the *method of dominant balance* of statistical mechanics [44,46]. This method is generally easier to apply than an analysis of the functional equation via Faa di Bruno's formula, and yields an algorithm for obtaining the amplitude recursion. In addition, the method allows one to analyse corrections to the limiting behaviour. The behaviour of the moments follows then by standard methods from singularity analysis of generating functions [22,24]. In Proposition 5.4, we give a quasi-linear partial differential equation for the generating function of the amplitudes. Growth estimates for the amplitudes (and hence for the moments) are obtained in Section 6, by an analysis of the (singular) partial differential equation. Existence and uniqueness of a limit distribution is then guaranteed by Lévy's continuity theorem, see Section 7. The connection to Brownian excursions follows in Section 8, by a comparison with Dyck paths. Possible further applications of our method are discussed in a concluding section.

2. q-functional equations

Let $\mathbb{C}[[\boldsymbol{u}]]$ denote the ring of formal power series with complex coefficients in the (commutative) variables $\boldsymbol{u} = (u_0, u_1, \ldots, u_M)$. Let $\mathbb{C}[\boldsymbol{u}]$ denote the ring of polynomials with complex coefficients in the variables $\boldsymbol{u}$. We set $\boldsymbol{u}_0 = (u_0, 1, \ldots, 1)$. For $\boldsymbol{u} = (u_0, u_1, \ldots, u_M)$ and $\boldsymbol{n} \in \mathbb{N}_0^{1+M}$, we define $\boldsymbol{u}^{\boldsymbol{n}}$ to be the monomial $\boldsymbol{u}^{\boldsymbol{n}} = u_0^{n_0} \cdot \ldots \cdot u_M^{n_M}$. We employ the notation $\boldsymbol{u}_+ = (u_1, \ldots, u_M)$, the plus sign indicating that the first component of $\boldsymbol{u}$ is omitted. For $n \in \mathbb{N}$ and $r > 0$, let D_r^n denote the open polydisc

$$D_r^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_k| < r \text{ for all } k = 1, \ldots, n\}.$$

Let $\mathcal{H}_r(\boldsymbol{x})$ denote the ring of power series in $\boldsymbol{x} = (x_1, \ldots, x_n)$ with complex coefficients, which are convergent in D_r^n.

Definition 2.1 (*q-shift*). Fix $M \in \mathbb{N}$. Let for $k = 0, \ldots, M$ formal power series $v_k(\boldsymbol{u}) \in \mathbb{C}[[\boldsymbol{u}]]$ be given, and write $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{u}) = (v_0(\boldsymbol{u}), v_1(\boldsymbol{u}), \ldots, v_M(\boldsymbol{u}))$. Assume that there is a number $d = d(\boldsymbol{v})$ satisfying $1 < d \leq \infty$, such that

$$v_0(\boldsymbol{u}) \in \mathcal{H}_d(u_1, \ldots, u_M)[[u_0]], \qquad v_k(\boldsymbol{u}) \in \mathcal{H}_d(u_k, \ldots, u_M) \quad (k = 1, \ldots, M).$$

Assume that $\boldsymbol{v}(\boldsymbol{u})$ satisfies

$$\boldsymbol{v}(\boldsymbol{u}_0) = \boldsymbol{u}_0, \qquad v_0(0, \boldsymbol{u}_+) \equiv 0, \qquad \frac{\partial v_k}{\partial u_k}(\boldsymbol{u}_0) \equiv 1 \quad (k = 0, \ldots, M).$$

Let $r = r(\boldsymbol{v})$ be a number $0 < r \leq \infty$ such that

$$\left\{(v_1(\boldsymbol{u}_+), \ldots, v_M(\boldsymbol{u}_+)) : \boldsymbol{u}_+ \in D_d^M\right\} \subseteq D_r^M.$$

Then, $\boldsymbol{v}$ is called a *q-shift*, $d(\boldsymbol{v})$ is called the *domain of* $\boldsymbol{v}$, and $r(\boldsymbol{v})$ is called the *range of* $\boldsymbol{v}$.

Remark. (i) For $M = 1$, a simple example of a q-shift appears in the q-difference equation (1.2), with $v_0(u_0, u_1) = u_0 u_1$ and $v_1(u_1) = u_1$. This motivates the name for the generalisation.

(ii) For a q-shift $\boldsymbol{v}$, its kth component $v_k(\boldsymbol{u})$ does not depend on u_l, where $l = 0, \ldots, k-1$, and it does depend on u_k. Since $\boldsymbol{v}(\boldsymbol{u}_0) = \boldsymbol{u}_0$, one may interpret $\boldsymbol{v}(\boldsymbol{u})$ for $\boldsymbol{u} \neq \boldsymbol{u}_0$ as a "deformation" of $\boldsymbol{u}_0$. The condition $v_0(0, \boldsymbol{u}_+) \equiv 0$ is imposed to ensure that composition of formal power series is well defined; see below.

(iii) The identity $id : \boldsymbol{u} \mapsto \boldsymbol{u}$ is a q-shift with infinite domain and range. For two q-shifts $\boldsymbol{v}$ and $\boldsymbol{w}$, their composition $\boldsymbol{v} \circ \boldsymbol{w}$ is well defined if $r(\boldsymbol{w}) \leq d(\boldsymbol{v})$. As is readily checked, $\boldsymbol{v} \circ \boldsymbol{w}$ is a q-shift in that case, with domain $d(\boldsymbol{v} \circ \boldsymbol{w}) = d(\boldsymbol{w})$ and range $r(\boldsymbol{v} \circ \boldsymbol{w}) = r(\boldsymbol{v})$. If for a q-shift $\boldsymbol{v}$ we have $r(\boldsymbol{v}) \leq d(\boldsymbol{v})$, we can thus consider *iterated* q-shifts $\boldsymbol{v}^{[n]}$, where

$$\boldsymbol{v}^{[0]} = id, \qquad \boldsymbol{v}^{[n]} = \boldsymbol{v} \circ \boldsymbol{v}^{[n-1]} \quad (n \in \mathbb{N}).$$

Then $\boldsymbol{v}^{[n]}$ is a q-shift for $n \in \mathbb{N}_0$.

(iv) An important subclass (with infinite domain and range) is *polynomial* q-shifts, i.e., q-shifts $\boldsymbol{v}$ satisfying $v_k(\boldsymbol{u}) \in \mathbb{C}[\boldsymbol{u}]$ for $k = 0, \ldots, M$. Examples are given by the monomial q-shifts

$$v_k(\boldsymbol{u}) = \boldsymbol{u}^{\boldsymbol{n}_k},$$

where $\boldsymbol{n}_k \in \mathbb{N}_0^{1+M}$, $(\boldsymbol{n}_k)_k = 1$ and $(\boldsymbol{n}_k)_l = 0$ for $l = 0, \ldots, k-1$ and $k = 0, \ldots, M$. These appear in Eqs. (1.2) and (1.3), and for Dyck paths in Eq. (1.12). In these examples, we have $(\boldsymbol{n}_l)_{l+1} \neq 0$ for $l = 0, \ldots, M-1$.

For a formal power series $G(\boldsymbol{u}) \in \mathcal{H}_d(\boldsymbol{u}_+)[[u_0]] \subseteq \mathbb{C}[[\boldsymbol{u}]]$ and a q-shift $\boldsymbol{v}$ satisfying $r(\boldsymbol{v}) \leq d$, we define $H(\boldsymbol{u}) \in \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)[[u_0]]$ by

$$H(\boldsymbol{u}) = G(\boldsymbol{v}(\boldsymbol{u})).$$

This is well defined since $\mathcal{H}_d(\boldsymbol{v}_+(\boldsymbol{u})) \subseteq \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)$ and $\mathbb{C}[[v_0(\boldsymbol{u})]] \subseteq \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)[[u_0]]$, due to $v_0(0, \boldsymbol{u}_+) \equiv 0$. We are interested in derivatives of $H(\boldsymbol{u})$. For the clarity of presentation, we will use the multi-index notation, and Greek indices $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\rho}$ will denote vectors with non-negative integer entries. For $F(\boldsymbol{u}) \in \mathbb{C}[[\boldsymbol{u}]]$ and $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_M) \in \mathbb{N}_0^{1+M}$, we write the derivative $F_{\boldsymbol{\nu}}(\boldsymbol{u}) \in \mathbb{C}[[\boldsymbol{u}]]$ of $F(\boldsymbol{u})$ of order $\boldsymbol{\nu}$ as

$$F_{\boldsymbol{\nu}}(\boldsymbol{u}) = \partial_0^{\nu_0} \cdots \partial_M^{\nu_M} F(\boldsymbol{u}), \qquad \partial_k^{\nu_k} = \frac{\partial^{\nu_k}}{\partial u_k^{\nu_k}} \quad (k = 0, \ldots, M),$$

where we use the convention that $F_{\boldsymbol{0}}(\boldsymbol{u}) = F(\boldsymbol{u})$. If $G(\boldsymbol{u}) \in \mathcal{H}_d(\boldsymbol{u}_+)[[u_0]] \subset \mathbb{C}[[\boldsymbol{u}]]$, we also have $G_{\boldsymbol{\nu}}(\boldsymbol{u}) \in \mathcal{H}_d(\boldsymbol{u}_+)[[u_0]]$. Set $|\boldsymbol{\nu}| = \nu_0 + \cdots + \nu_M$ and $\boldsymbol{\nu}! = \nu_0! \cdot \cdots \cdot \nu_M!$. We define a total order in $\mathbb{N}_0^{1+M}$ as follows.

Definition 2.2 (*Total Order* $\prec$). For $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_M) \in \mathbb{N}_0^{1+M}$ and $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_M) \in \mathbb{N}_0^{1+M}$, we write $\boldsymbol{\mu} \prec \boldsymbol{\nu}$ if either $|\boldsymbol{\mu}| < |\boldsymbol{\nu}|$, or if $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$, there exists an index $k \in \{0, \ldots, M\}$, such that $\mu_k > \nu_k$ and $\mu_i = \nu_i$ for $i = 0, \ldots, k-1$.

The total order introduced above will be used to label terms appearing in derivatives of $H(\boldsymbol{u})$. We have the following lemma. For $i \in \{0, \ldots, M\}$, let $\boldsymbol{e}_i \in \mathbb{C}^{1+M}$ denote the unit vector in direction i, given by $(\boldsymbol{e}_i)_k = \delta_{i,k}$ for $k = 0, \ldots, M$.

Lemma 2.3. *Let $G(\boldsymbol{u}) \in \mathcal{H}_d(\boldsymbol{u}_+)[[u_0]]$, and let a q-shift $\boldsymbol{v}$ with domain $d(\boldsymbol{v})$ and range $r(\boldsymbol{v}) \leq d$ be given. Set $H(\boldsymbol{u}) = G(\boldsymbol{v}(\boldsymbol{u})) \in \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)[[u_0]]$ and fix $\boldsymbol{\nu} \neq \boldsymbol{0}$.*

(i) *We have $H_{\boldsymbol{\nu}}(\boldsymbol{u}) \in \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)[[u_0]]$. For every $\boldsymbol{\mu}$ satisfying $\boldsymbol{0} \neq \boldsymbol{\mu} \preceq \boldsymbol{\nu}$, there exists a coefficient $A(\boldsymbol{\mu}, \boldsymbol{u}) \in \mathcal{H}_{d(\boldsymbol{v})}(\boldsymbol{u}_+)[[u_0]]$, independent of the choice of $G(\boldsymbol{u})$, such that*

$$H_{\boldsymbol{\nu}}(\boldsymbol{u}) = \sum_{\boldsymbol{0} \neq \boldsymbol{\mu} \preceq \boldsymbol{\nu}} G_{\boldsymbol{\mu}}(\boldsymbol{v}(\boldsymbol{u})) \cdot A(\boldsymbol{\mu}, \boldsymbol{u}). \tag{2.1}$$

(ii) *The coefficient $A(\boldsymbol{\mu}, \boldsymbol{u})$ in Eq. (2.1) is, for $\boldsymbol{\mu} = \boldsymbol{\nu}$, given by*

$$A(\boldsymbol{\nu}, \boldsymbol{u}) = \prod_{k=0}^{M} \left(\frac{\partial v_k}{\partial u_k}(\boldsymbol{u}) \right)^{\nu_k}. \tag{2.2}$$

In particular, we have $A(\boldsymbol{\nu}, \boldsymbol{u}_0) = 1$.

(iii) *Fix $i \in \{0, \ldots, M-1\}$. If $\nu_{i+1} > 0$, we have $\boldsymbol{\nu} - \boldsymbol{e}_{i+1} + \boldsymbol{e}_i \prec \boldsymbol{\nu}$. The coefficient $A(\boldsymbol{\mu}, \boldsymbol{u})$ in Eq. (2.1) is, for $\boldsymbol{\mu} = \boldsymbol{\nu} - \boldsymbol{e}_{i+1} + \boldsymbol{e}_i$ and $\boldsymbol{u} = \boldsymbol{u}_0$, given by*

$$A(\boldsymbol{\nu} - \boldsymbol{e}_{i+1} + \boldsymbol{e}_i, \boldsymbol{u}_0) = \nu_{i+1} \frac{\partial v_i}{\partial u_{i+1}}(\boldsymbol{u}_0).$$

(iv) *For real numbers $r, r_0, \ldots, r_M$, where $r_{k+1} > r_k$ for $k = 0, \ldots, M-1$, define $\alpha_{\boldsymbol{\mu}} = r + \sum_{k=0}^{M} r_k \mu_k$. Then, for indices $\boldsymbol{\mu} \prec \boldsymbol{\nu}$ in Eq. (2.1) satisfying $A(\boldsymbol{\mu}, \boldsymbol{u}) \not\equiv 0$, we have $\alpha_{\boldsymbol{\mu}} < \alpha_{\boldsymbol{\nu}}$.*

Remark. (i) The coefficient $A(\boldsymbol{\mu}, \boldsymbol{u})$ in Eq. (2.1) may be chosen to vanish. For example, fix $\boldsymbol{\nu} = (1, 1)$ and consider $\boldsymbol{\mu} = (0, 1)$. We have $\boldsymbol{\mu} \prec \boldsymbol{\nu}$ but might choose $A(\boldsymbol{\mu}, \boldsymbol{u}) \equiv 0$, as is readily verified by an explicit calculation, using the q-shift property $v_k(\boldsymbol{u}) \in \mathbb{C}[[u_k, \ldots, u_M]]$.

(ii) The property (iv) will be used for exponent estimates in Proposition 4.4.

Proof. (i) This is an application of the chain rule. Successive differentiation leads to

$$H_{\boldsymbol{\nu}}(\boldsymbol{u}) = \sum_{0 \leq |\boldsymbol{\mu}| \leq |\boldsymbol{\nu}|} G_{\boldsymbol{\mu}}(\boldsymbol{v}(\boldsymbol{u})) \cdot A(\boldsymbol{\mu}, \boldsymbol{u}), \tag{2.3}$$

where $A(\boldsymbol{\mu}, \boldsymbol{u})$ is independent of $G(\boldsymbol{u})$. To analyse the effect of the q-shift property $v_k(\boldsymbol{u}) \in \mathcal{H}_{d(\boldsymbol{v})}(u_k, \ldots, u_M)$, consider for $k \in \{0, \ldots, M\}$ the equation

$$\frac{\partial H}{\partial u_k}(\boldsymbol{u}) = \sum_{l=0}^{M} \frac{\partial G}{\partial v_l}(\boldsymbol{v}(\boldsymbol{u})) \frac{\partial v_l}{\partial u_k}(\boldsymbol{u}) = \sum_{l=0}^{k} \frac{\partial G}{\partial v_l}(\boldsymbol{v}(\boldsymbol{u})) \frac{\partial v_l}{\partial u_k}(\boldsymbol{u}). \tag{2.4}$$

It shows that derivatives of $H(\boldsymbol{u})$ w.r.t. u_k do not contribute to derivatives of $G(\boldsymbol{v}(\boldsymbol{u}))$ w.r.t. $v_{k+1}, \ldots, v_M$. A similar statement holds for higher derivatives. This implies that the numbers ν_k in Eq. (2.3) can contribute to the numbers $\mu_0, \ldots, \mu_k$ only. If $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$, we thus have $\nu_k \le \mu_0 + \cdots + \mu_k$ for $k \in \{0, \ldots, M\}$. We show that $\boldsymbol{\mu} \preceq \boldsymbol{\nu}$. Assume w.l.o.g. that $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$. If $\mu_0 > \nu_0$, we have $\boldsymbol{\mu} \prec \boldsymbol{\nu}$, and the claim follows. Otherwise, we have $\mu_0 = \nu_0$. Thus, ν_k does not contribute to μ_0 for $k \in \{1, \ldots, M\}$. The previous argument yields that $\nu_k \le \mu_1 + \cdots + \mu_k$ for $k \in \{1, \ldots, M\}$. Now repeat the above argument until $\mu_0 = \cdots = \mu_{M-1} = \nu_0 = \cdots = \nu_{M-1}$. Then $\mu_M = \nu_M$ since $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$. Thus $\boldsymbol{\mu} = \boldsymbol{\nu}$, and the statement is shown.

(ii) This is seen by an explicit calculation using Eq. (2.4). Successively applying derivatives w.r.t. $u_0, u_1, \ldots, u_M$ yields Eq. (2.2). Together with the q-shift property $\partial v_k / \partial u_k(\boldsymbol{u}_0) = 1$, it follows that $A(\boldsymbol{\nu}, \boldsymbol{u}_0) = 1$. Statement (iii) is shown by an analogous calculation.

(iv) Consider first the case $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$. Note that, by differentiation, the numbers ν_k do not contribute to $\mu_{k+1}, \ldots, \mu_M$, for $k \in \{0, \ldots, M\}$. Thus, the contribution of ν_k to $\alpha_{\boldsymbol{\mu}}$ is maximal if $\mu_k = \nu_k$ for $k = 0, \ldots, M$. We get $\boldsymbol{\mu} = \boldsymbol{\nu}$ and $\alpha_{\boldsymbol{\mu}} = \alpha_{\boldsymbol{\nu}}$. Since this maximum is unique, $\boldsymbol{\mu} \prec \boldsymbol{\nu}$ implies $\alpha_{\boldsymbol{\mu}} < \alpha_{\boldsymbol{\nu}}$. Now let $|\boldsymbol{\mu}| < |\boldsymbol{\nu}|$. For $k \in \{0, \ldots, M\}$, denote by $\widetilde{\nu}_k$ the number of derivatives w.r.t. u_k, which contribute to $\boldsymbol{\mu}$. Set $\widetilde{\boldsymbol{\nu}} = (\widetilde{\nu}_0, \ldots, \widetilde{\nu}_M)$. Then clearly $|\boldsymbol{\mu}| = |\widetilde{\boldsymbol{\nu}}|$ and $\alpha_{\widetilde{\boldsymbol{\nu}}} < \alpha_{\boldsymbol{\nu}}$. The reasoning for the case $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$ can now be applied to the present case, with $\boldsymbol{\nu}$ replaced by $\widetilde{\boldsymbol{\nu}}$. This yields $\alpha_{\boldsymbol{\mu}} \le \alpha_{\widetilde{\boldsymbol{\nu}}}$. Thus $\alpha_{\boldsymbol{\mu}} < \alpha_{\boldsymbol{\nu}}$, and the statement is shown. □

Definition 2.4 (*q-Functional Equation*). Let $P(\boldsymbol{u}, y_1, \ldots, y_N)$ be a formal power series $P(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{H}_d(\boldsymbol{u}_+)[[u_0, \boldsymbol{y}]]$, for a number d satisfying $1 < d \le \infty$. Assume that $P(0, \boldsymbol{u}_+, \boldsymbol{0}) \equiv 0$, and that $\frac{\partial P}{\partial y_j}(0, \boldsymbol{u}_+, \boldsymbol{0}) \equiv 0$ for $j = 1, \ldots, N$. Let $\boldsymbol{v}^{(j)}$ be a q-shift with domain $d(\boldsymbol{v}^{(j)})$ and range $r(\boldsymbol{v}^{(j)}) \le d$, for $j = 1, \ldots, N$.

(i) If the above assumptions are satisfied, the equation

$$G(\boldsymbol{u}) = P(\boldsymbol{u}, H^{(1)}(\boldsymbol{u}), \ldots, H^{(N)}(\boldsymbol{u})), \tag{2.5}$$

where $H^{(j)}(\boldsymbol{u}) = G(\boldsymbol{v}^{(j)}(\boldsymbol{u}))$ for $j = 1, \ldots, N$, is called a *q-functional equation*.

(ii) Let the above assumptions be satisfied. If there is a number $d(G)$ such that

$$1 < d(G) \le \min_{1 \le j \le N} \{d(\boldsymbol{v}^{(j)})\},$$

and a formal power series $G(\boldsymbol{u}) \in \mathcal{H}_{d(G)}(\boldsymbol{u}_+)[[u_0]]$ satisfying Eq. (2.5), then $G(\boldsymbol{u})$ is called a *solution* of the q-functional equation.

Remark. (i) An example of a q-functional equation for $M = 1$ is given by the q-difference equation (Eq. (1.2)). This motivates the name for the generalisation. Examples for $M > 1$ appear in Eq. (1.3). Examples of q-functional equations frequently satisfy $P(\boldsymbol{u}, \boldsymbol{y}) \in \mathbb{C}[\boldsymbol{u}, \boldsymbol{y}]$ with $d = \infty$, and with monomial q-shifts. We infer from the functional equation (1.12) for Dyck paths that the power series $E(\boldsymbol{u}) = D(\boldsymbol{u}) - 1$ satisfies the q-functional equation

$$E(\boldsymbol{u}) = u_0 u_1 \cdot \ldots \cdot u_M \left(E(\boldsymbol{u}) + 1\right) \left(E(\boldsymbol{v}(\boldsymbol{u})) + 1\right). \tag{2.6}$$

Specialising to $\boldsymbol{u} = \boldsymbol{u}_0$ yields a quadratic equation, which can be explicitly solved for $E(\boldsymbol{u}_0)$. We get the well-known result

$$E(\boldsymbol{u}_0) = \frac{1 - 2u_0 - \sqrt{1 - 4u_0}}{2u_0}.$$

(ii) If $Q(u, y) \in \mathbb{C}[[u, y]]$ satisfies $Q(0, 0) = 0$ and $\frac{\partial Q}{\partial y}(0, y) \not\equiv 1$, a formal power series $G(u) \in \mathbb{C}[[u]]$ such that $G(0) = 0$ is uniquely defined as the solution of the equation $G(u) = Q(u, G(u))$. When studying such equations, we may assume without loss of generality that $\frac{\partial Q}{\partial y}(0, y) \equiv 0$. The above definition reduces to this setup, when restricted to $\boldsymbol{u} = \boldsymbol{u}_0$.

A result about solutions of q-functional equations is given by the following proposition. We use the vector notation $\boldsymbol{H}(\boldsymbol{u}) = (H^{(1)}(\boldsymbol{u}), \ldots, H^{(N)}(\boldsymbol{u}))$.

Proposition 2.5. *The q-functional equation of Definition 2.4,*

$$G(\boldsymbol{u}) = P(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u})), \tag{2.7}$$

has a unique solution $G(\boldsymbol{u}) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$ satisfying $G(0, \boldsymbol{u}_+) \equiv 0$, where $d(P) = \min_{1\le j\le N}\{d(\boldsymbol{v}^{(j)})\}$.

Proof. If $G(\boldsymbol{u}) = \sum_{n_0\ge 1} p_{n_0}(\boldsymbol{u}_+)u_0^{n_0}$, then $G(\boldsymbol{v}^{(j)}(\boldsymbol{u})) = \sum_{n_0\ge 1} p_{n_0}(\boldsymbol{v}_+^{(j)}(\boldsymbol{u}))\left(v_0^{(j)}(\boldsymbol{u})\right)^{n_0}$. For $n_0 \ge 1$ fixed, we take the coefficient of $u_0^{n_0}$ in Eq. (2.7). Due to $v_0^{(j)}(0, \boldsymbol{u}_+) \equiv 0$ for $j = 1, \ldots, N$ and the assumptions on the derivatives of $P(\boldsymbol{u}, \boldsymbol{y})$, we get the expression

$$p_{n_0}(\boldsymbol{u}_+) = W_{n_0}\left(\boldsymbol{u}_+, \left\{p_1(\boldsymbol{v}_+^{(j)}(\boldsymbol{u})), \ldots, p_{n_0-1}(\boldsymbol{v}_+^{(j)}(\boldsymbol{u}))\right\}_{j=1}^{N}\right), \tag{2.8}$$

for a power series $W_{n_0}(\boldsymbol{u}_+, \boldsymbol{p}) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[\boldsymbol{p}]$ in $M + N(n_0 - 1)$ variables. Thus $p_{n_0}(\boldsymbol{u}_+)$ is determined recursively in terms of $p_l(\boldsymbol{u}_+)$, where $l = 1, \ldots, n_0 - 1$. The recursion also shows that $p_{n_0}(\boldsymbol{u}_+) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)$. Thus $G(\boldsymbol{u}) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$, and the proposition is proved. □

Remark. (i) For a solution $G(\boldsymbol{u})$ of a q-functional equation satisfying $G(\boldsymbol{0}) = 0$, we will always assume $d(G) = d(P)$ in the following.

(ii) For the solution of a simple q-functional equation, an explicit expression may be given. This is e.g. the case for some q-difference equations Eq. (1.2), with $P(x, y)$ linear in y, see e.g. [9], or with $P(x, y)$ quadratic in y, see [44].

(iii) It follows from $G(\boldsymbol{u}) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$ that $G(\boldsymbol{u}_0) \in \mathbb{C}[[u_0]]$. For derivatives of $G(\boldsymbol{u})$, which are also elements of $\mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$, the same conclusion holds. Thus, the derivatives

$$g_{\boldsymbol{\nu}}(u_0) := \left.\frac{1}{\boldsymbol{\nu}!}G_{\boldsymbol{\nu}}(\boldsymbol{u})\right|_{\boldsymbol{u}=\boldsymbol{u}_0} \tag{2.9}$$

are formal power series, i.e., $g_{\boldsymbol{\nu}}(u_0) \in \mathbb{C}[[u_0]]$. They are called *factorial moment generating functions*, for reasons to be explained in Section 7.

3. Factorial moment generating functions

Due to the following proposition, the factorial moment generating functions can be computed recursively, by successively differentiating the q-functional equation.

Proposition 3.1. *Let a q-functional equation (Eq. (2.5)) with solution $G(\boldsymbol{u})$ as in Proposition 2.5 be given. Consider the derivative of order $\boldsymbol{\nu} \ne \boldsymbol{0}$ of the q-functional equation, evaluated at $\boldsymbol{u} = \boldsymbol{u}_0$. It is linear in $g_{\boldsymbol{\nu}}(u_0)$. Its r.h.s. is a polynomial in $\{g_{\boldsymbol{\mu}}(u_0) : \boldsymbol{0} \ne \boldsymbol{\mu} \preceq \boldsymbol{\nu}\}$, with coefficients in $\mathbb{C}[[u_0, g_{\boldsymbol{0}}(u_0)]]$.*

In order to prove Proposition 3.1, we analyse partial derivatives of Eq. (2.5). To this end, we employ a generalisation of Faa di Bruno's formula [15], adapted to our situation. The following lemma will also be used for exponent estimates in the next section. For $k \in \{1, \ldots, K\}$, let $f_k(\boldsymbol{u}) \in \mathbb{C}[[\boldsymbol{u}]]$, and define $\boldsymbol{f}(\boldsymbol{u}) = (f_1(\boldsymbol{u}), \ldots, f_K(\boldsymbol{u}))$. For $\boldsymbol{\mu} \in \mathbb{N}_0^{1+M}$, we use the notation $\boldsymbol{f}(\boldsymbol{u})_{\boldsymbol{\mu}} = ((f_1)_{\boldsymbol{\mu}}(\boldsymbol{u}), \ldots, (f_K)_{\boldsymbol{\mu}}(\boldsymbol{u}))$ for the derivative of $\boldsymbol{f}(\boldsymbol{u})$ of order $\boldsymbol{\mu}$.

Lemma 3.2 (*cf. [15, Thm. 2.1]*). *Let a q-functional equation (2.5) with solution $G(\boldsymbol{u})$ as in Proposition 2.5 be given. Its derivative of order $\boldsymbol{\nu} \ne \boldsymbol{0}$ satisfies*

$$G_{\boldsymbol{\nu}}(\boldsymbol{u}) = \sum_{1\le|\boldsymbol{\lambda}|\le|\boldsymbol{\nu}|} P_{\boldsymbol{\lambda}}(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u}))\sum_{s=1}^{|\boldsymbol{\nu}|}\sum_{p_s(\boldsymbol{\nu},\boldsymbol{\lambda})}(\boldsymbol{\nu}!)\prod_{j=1}^{s}\frac{[(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u}))_{\boldsymbol{\mu}_j}]^{\boldsymbol{\kappa}_j}}{(\boldsymbol{\kappa}_j!)[\boldsymbol{\mu}_j!]^{|\boldsymbol{\kappa}_j|}}, \tag{3.1}$$

where the vectors $\boldsymbol{\lambda}$ *and* $\boldsymbol{\kappa}_j$ *are* $1+M+N$*-dimensional, the vectors* $\boldsymbol{\mu}_j$ *are* $1+M$*-dimensional, for* $j \in \{1, \ldots, s\}$*, and the summation ranges over*

$$p_s(\boldsymbol{\nu}, \boldsymbol{\lambda}) = \left\{ (\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_s; \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_s) : |\boldsymbol{\kappa}_i| > 0, \mathbf{0} \lhd \boldsymbol{\mu}_1 \lhd \cdots \lhd \boldsymbol{\mu}_s, \sum_{i=1}^{s} \boldsymbol{\kappa}_i = \boldsymbol{\lambda} \text{ and } \sum_{i=1}^{s} |\boldsymbol{\kappa}_i| \boldsymbol{\mu}_i = \boldsymbol{\nu} \right\}.$$

In the above equation, the total order $\lhd$ *is defined as* $\boldsymbol{\mu} \lhd \boldsymbol{\nu}$ *if either* $|\boldsymbol{\mu}| < |\boldsymbol{\nu}|$*, or if* $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$*, there exists an index* $k \in \{0, \ldots, M\}$ *such that* $\mu_k < \nu_k$ *and* $\mu_i = \nu_i$ *for* $i = 0, \ldots, k-1$. □

Remark. If $M = 1$, we have $(\mu_0, \mu_1) \lhd (\nu_0, \nu_1)$ if and only if $(\mu_1, \mu_0) \prec (\nu_1, \nu_0)$. The analogous statement for higher values of M is not true.

Proof (*Proof of* Proposition 3.1.). For given $\boldsymbol{\nu} \neq \mathbf{0}$, we analyse the values $\boldsymbol{\mu}_j$ appearing in Lemma 3.2. We show that $|\boldsymbol{\mu}_j| \geq |\boldsymbol{\nu}|$ for some value j, where $1 \leq j \leq s \leq |\boldsymbol{\nu}|$, implies $s = 1$ and $\boldsymbol{\mu}_1 = \boldsymbol{\nu}$. Together with Lemma 2.3, this implies that $\boldsymbol{\mu} \preceq \boldsymbol{\nu}$. Linearity will follow from an explicit calculation of the term containing $\boldsymbol{\mu}_1 = \boldsymbol{\nu}$.

Assume that $|\boldsymbol{\mu}_j| \geq |\boldsymbol{\nu}|$ for some j, where $1 \leq j \leq s \leq |\boldsymbol{\nu}|$. The explicit form of $p_s(\boldsymbol{\nu}, \boldsymbol{\lambda})$ in Lemma 3.2 states that $\boldsymbol{\nu} = \sum_{i=1}^{s} |\boldsymbol{\kappa}_i| \boldsymbol{\mu}_i$. This implies that $|\boldsymbol{\nu}| = \sum_{i=1}^{s} |\boldsymbol{\kappa}_i||\boldsymbol{\mu}_i|$. According to the assumption, this leads to $s = 1$, $|\boldsymbol{\kappa}_1| = 1$ and $|\boldsymbol{\mu}_1| = |\boldsymbol{\nu}|$. This, in turn, implies that $\boldsymbol{\mu}_1 = \boldsymbol{\nu}$.

Clearly, the r.h.s. of Eq. (3.1), when specialised to $\boldsymbol{u} = \boldsymbol{u}_0$, is a polynomial in $g_{\boldsymbol{\mu}}(u_0)$ for $\mathbf{0} \neq \boldsymbol{\mu} \preceq \boldsymbol{\nu}$, with coefficients in $\mathbb{C}[[u_0, g_{\mathbf{0}}(u_0)]]$. To show linearity in $\boldsymbol{G}_{\boldsymbol{\nu}}(\boldsymbol{u}_0)$, we note that the possible values of $\boldsymbol{\kappa}_1$ are $\boldsymbol{\kappa}_1 = \boldsymbol{e}_j$, where $j \in \{1, \ldots, 1+M+N\}$. The sum of the terms with $|\boldsymbol{\mu}_1| = |\boldsymbol{\nu}|$ in the r.h.s. of Eq. (3.1) is given by

$$\sum_{j=1}^{1+M+N} P_{\boldsymbol{e}_j}(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u}))[(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u}))_{\boldsymbol{\nu}}]^{\boldsymbol{e}_j}.$$

We now extract terms containing $\boldsymbol{G}_{\boldsymbol{\nu}}(\boldsymbol{u})$ from this expression, using Lemma 2.3, and group them to the l.h.s. of Eq. (3.1). In the resulting equation, the l.h.s. $L(\boldsymbol{u})$, when specialised to $\boldsymbol{u} = \boldsymbol{u}_0$, is given by

$$L(\boldsymbol{u}_0) = \left(1 - \sum_{j=1}^{N} \frac{\partial P}{\partial y_j}(\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0)) \right) \boldsymbol{G}_{\boldsymbol{\nu}}(\boldsymbol{u}_0). \tag{3.2}$$

Due to the assumptions on $P(\boldsymbol{u}, \boldsymbol{y})$, the prefactor of $\boldsymbol{G}_{\boldsymbol{\nu}}(\boldsymbol{u}_0)$ in the above equation is not identically vanishing. Thus, $\boldsymbol{G}_{\boldsymbol{\nu}}(\boldsymbol{u}_0)$ is contained linearly in Eq. (3.1), specialised to $\boldsymbol{u} = \boldsymbol{u}_0$. □

4. Analytic generating functions

Let $Q(u, y) = P(u, 1, \ldots, 1, y, \ldots, y)$. The power series $G(\boldsymbol{u}_0)$ satisfies the equation

$$G(\boldsymbol{u}_0) = Q(u_0, G(\boldsymbol{u}_0)). \tag{4.1}$$

In the following, we specialise the class of q-functional equations. We are interested in the case where $G(\boldsymbol{u}_0)$ is analytic at $u_0 = 0$, with a square root as dominant singularity. This situation is generic for combinatorial constructions, see [41, Thm. 10.6], [16, Prop. 1], or [24, Sec. 7.4]. Throughout the remainder of the article, we employ the following assumption.

Assumption 4.1. Let a q-functional equation Eq. (2.5) as in Definition 2.4 be given. Let numbers r, s, such that $0 < r, s \leq \infty$, be given. Denote by $\mathcal{H}_{r,d,s}(\boldsymbol{u}, \boldsymbol{y})$ the ring of power series in $(\boldsymbol{u}, \boldsymbol{y}) = (u_0, \boldsymbol{u}_+, \boldsymbol{y})$, which are convergent if $|u_0| < r$, if $|u_k| < d$ for $k = 1, \ldots, M$, and if $|y_k| < s$ for $k = 1, \ldots, N$. In addition to the assumptions in Definition 2.4, we assume the following properties.

(i) $P(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{H}_{r,d,s}(\boldsymbol{u}, \boldsymbol{y})$, and its Taylor coefficients are non-negative real numbers, when expanded about $(\boldsymbol{u}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$. For $j = 1, \ldots, N$, each coordinate series of the q-shift $\boldsymbol{v}^{(j)}(\boldsymbol{u})$ has non-negative coefficients only, if expanded about $\boldsymbol{u} = \mathbf{0}$.

(ii) Let $Q(u, y) = P(u, 1, \dots, 1, y, \dots, y)$. There exist numbers (u_c, y_c) satisfying $0 < u_c < r$ and $0 < y_c < s$, such that

$$\begin{aligned} &Q(u_c, y_c) = y_c, \qquad \left.\frac{\partial Q}{\partial y}(u_c, y)\right|_{y=y_c} = 1, \\ &B := \frac{1}{2}\left.\frac{\partial^2 Q}{\partial y^2}(u_c, y)\right|_{y=y_c} > 0, \qquad C := \left.\frac{\partial Q}{\partial u}(u, y_c)\right|_{u=u_c} > 0. \end{aligned} \tag{4.2}$$

(iii) The solution $G(\boldsymbol{u})$ of the q-functional equation as in Proposition 2.5 has the property that $G(\boldsymbol{u}_0) = \sum_{n\geq 1} p_n u_0^n$ is *aperiodic*, i.e., there exist indices $1 \leq i < j < k$ such that $p_i p_j p_k \neq 0$, while $\gcd(j - i, k - i) = 1$.

Remark. (i) When restricted to $\boldsymbol{u} = \boldsymbol{u}_0$, the above assumptions (together with the assumptions in Definition 2.4) reduce to the setup for implicitly defined power series which is usual in enumerative combinatorics, see [41, Thm. 10.6], [16, Prop. 1], and [24, Sec. 7.4].

(ii) Assumption 4.1 (i) implies that the coefficients $p_{\boldsymbol{n}}$ in $G(\boldsymbol{u}) = \sum_{\boldsymbol{n}\geq 0} p_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$ are non-negative. For combinatorial constructions, such positivity assumptions are common. However, systems of functional equations, arising from a combinatorial construction with positive coefficients, might be reduced to an equation of the above form having *negative* coefficients. In that situation, other types of singularity might appear. This is, e.g., the case for discrete meanders [39]. The methods used in this paper can be adapted to treat such cases. $P(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{H}_{r,d,s}(\boldsymbol{u}, \boldsymbol{y})$ implies that, for each $(\boldsymbol{u}, \boldsymbol{y}) \in D_r^1 \times D_d^M \times D_s^N$, the function $P(\boldsymbol{u}, \boldsymbol{y})$ has a convergent series expansion about $(\boldsymbol{u}, \boldsymbol{y})$.

(iii) Assumption 4.1 (ii) implies that the dominant singularity of $G(\boldsymbol{u}_0)$ is a square root. This type of singularity is generic for functional equations with positive coefficients. If $P(\boldsymbol{u}, \boldsymbol{y})$ is a *polynomial* in $\boldsymbol{u}$ and $\boldsymbol{y}$, it can be shown that Assumption 4.1(ii) follows from Assumption 4.1(i), if $Q(u, y)$ is of degree ≥ 2 in y and if $Q(u, 0) \not\equiv 0$. By the closure properties of algebraic functions, it then follows, with an adaption of Proposition 3.1, that all factorial moment generating functions are algebraic.

(iv) Assumption 4.1 (iii) ensures that $G(\boldsymbol{u}_0)$ has exactly one singularity on its circle of convergence. The case of periodic $G(\boldsymbol{u}_0)$ may be treated by a slight extension of this setup.

We investigate analytic properties of $g_{\boldsymbol{0}}(u_0)$. A function $f(u)$ is called *Δ-regular* [20] if it is analytic in the *indented disc* $\Delta = \Delta(u_c, \eta, \phi) = \{u : |u| \leq u_c + \eta, |\arg(u - u_c)| \geq \phi\}$ for some real numbers $u_c > 0$, $\eta > 0$ and ϕ, where $0 < \phi < \pi/2$. Note that $u_c \notin \Delta$, where we employ the convention $\arg(0) = 0$. The set of Δ-regular functions is closed under addition, multiplication, differentiation, and integration. Moreover, if $f(u) \neq 0$ in Δ, then $1/f(u)$ exists in Δ and is Δ-regular. The following proposition is a straightforward extension of a well-known result, see e.g. [41, Thm. 10.6], [16, Prop. 1], or [24, Sec. 7.4].

Proposition 4.2 (*cf. [41,16,24]*). *Given Assumption 4.1, the power series $g_{\boldsymbol{0}}(u_0)$ is analytic at $u_0 = 0$, with radius of convergence $0 < u_c < \infty$. Its analytic continuation is Δ-regular, with a square-root singularity at $u_0 = u_c$, and a local Puiseux expansion*

$$g_{\boldsymbol{0}}(u_0) = g_{\boldsymbol{0}}(u_c) + \sum_{l=0}^{\infty} f_{\boldsymbol{0},l}(u_c - u_0)^{-\gamma_{\boldsymbol{0}}+l/2}, \tag{4.3}$$

where $\gamma_{\boldsymbol{0}} = -1/2$, $f_{\boldsymbol{0},0} = -\sqrt{C/B}$, and $g_{\boldsymbol{0}}(u_c) = \lim_{u_0\to u_c^-} g_{\boldsymbol{0}}(u_0) < \infty$. $\square$

Remark. The coefficients $f_{\boldsymbol{0},l}$, also called *amplitudes*, can be computed recursively from the functional equation (4.1), by inserting the representation Eq. (4.3) into Eq. (4.1) and then expanding the resulting equation in $s = \sqrt{u_c - u_0}$. This technique will be exploited below, when using the method of dominant balance.

The properties of $G(\boldsymbol{u}_0)$ carry over to the factorial moment generating functions $g_{\boldsymbol{\nu}}(u_0)$. We will first analyse the general form of the factorial moment generating functions, and later provide explicit values for exponents and leading amplitudes.

Proposition 4.3. *Let Assumption 4.1 be satisfied. For $\boldsymbol{\nu} \neq \boldsymbol{0}$ arbitrary, the factorial moment generating function $g_{\boldsymbol{\nu}}(u_0)$ is analytic at $u_0 = 0$, with radius of convergence $0 < u_c < \infty$. Its analytic continuation is Δ-regular. It has a* *local Puiseux expansion*

$$g_{\boldsymbol{\nu}}(u_0) = \sum_{l=0}^{\infty} f_{\boldsymbol{\nu},l}(u_c - u_0)^{-\gamma_{\boldsymbol{\nu}}+l/2}, \tag{4.4}$$

with non-vanishing leading amplitude $f_{\boldsymbol{\nu},0} \neq 0$ *and exponent* $\gamma_{\boldsymbol{\nu}} \in \frac{1}{2}\mathbb{Z}$.

Proof. We prove the proposition by induction on $\boldsymbol{\nu}$ w.r.t. the total order $\prec$ in Definition 2.2. Note that the proof of Proposition 3.1 yields

$$g_{\boldsymbol{\nu}}(u_0) Q_1(u_0, g_{\mathbf{0}}(u_0)) = Q_{\boldsymbol{\nu}}(u_0, g_{\mathbf{0}}(u_0), \{g_{\boldsymbol{\mu}}(u_0)\}_{\mathbf{0}\neq\boldsymbol{\mu}\prec\boldsymbol{\nu}}), \tag{4.5}$$

where $Q_1(u, y) = 1 - \frac{\partial Q}{\partial y}(u, y)$ is analytic for $|u| < r$, $|y| < s$. The function $Q_{\boldsymbol{\nu}}(u, y, \widetilde{\boldsymbol{y}})$ is a polynomial in $\widetilde{\boldsymbol{y}}$ and analytic for $|u| < r$, $|y| < s$. Due to the closure properties of Δ-regular and analytic functions, both $Q_1(u_0, g_{\mathbf{0}}(u_0))$ and $Q_{\boldsymbol{\nu}}(u_0, g_{\mathbf{0}}(u_0), \{g_{\boldsymbol{\mu}}(u_0)\}_{\mathbf{0}\neq\boldsymbol{\mu}\prec\boldsymbol{\nu}})$ are Δ-regular. Due to Proposition 4.2, we have $Q_1(u_0, g_{\mathbf{0}}(u_0)) \neq 0$ in Δ, thus its inverse is Δ-regular. This implies that $g_{\boldsymbol{\nu}}(u_0)$ is Δ-regular.

Due to Proposition 4.2, the series $h(s) = y_c + \sum_{l=0}^{\infty} f_{\mathbf{0},l} s^{(l+1)}$ is holomorphic at $s = 0$ and equals $g_{\mathbf{0}}(u_0)$ about $u_0 = u_c$, if $s = \sqrt{u_c - u_0}$. We show that Eq. (4.5) leads to local expansions of $g_{\boldsymbol{\nu}}(u_0)$ in terms of $s = \sqrt{u_c - u_0}$, which are meromorphic at $s = 0$. Consider first the term $Q_1(u_0, g_{\mathbf{0}}(u_0))$. The function $Q_1(u, y)$ is holomorphic at $(u, y) = (u_c, y_c)$. Thus, as composition of holomorphic functions, $\widetilde{Q}_1(s) := Q_1(u_c - s^2, h(s))$ is holomorphic at $s = 0$. We have $\widetilde{Q}_1(0) = 0$ and $\widetilde{Q}_1'(0) = -\frac{\partial Q_1}{\partial y}(u_c, y_c)h'(0) \neq 0$. Thus, its inverse $1/\widetilde{Q}_1(s)$ is meromorphic at $s = 0$, with a simple pole. Consider finally the function $Q_{\boldsymbol{\nu}}(u, y, \widetilde{\boldsymbol{y}})$. It is a polynomial in $\widetilde{\boldsymbol{y}}$ and analytic at $(u, y) = (u_c, y_c)$. Thus, after inserting the expansions of $g_{\boldsymbol{\mu}}(u_0)$, where we use Proposition 4.2 and the induction hypothesis, we conclude that it is meromorphic in s at $s = 0$. It follows that $g_{\boldsymbol{\nu}}(u_0)$ is meromorphic in s at $s = 0$. Since $G(\boldsymbol{u})$ is non-negative and $g_{\mathbf{0}}(u_0)$ does not vanish identically, $g_{\boldsymbol{\nu}}(u_0)$ does not vanish identically. We thus have local expansions Eq. (4.4) of $g_{\boldsymbol{\nu}}(u_0)$. □

Remark. (i) It was argued in the preceding proof that $1 - \sum_{j=1}^{N} \frac{\partial P}{\partial y_j}(\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0)) = \widetilde{Q}_1(s)$ satisfies $\widetilde{Q}_1(s) \sim A\sqrt{u_c - u_0}$ as $u_0 \to u_c^-$, for some constant $A \neq 0$, see also Eq. (3.2). This will be used in the proof of the following proposition.

(ii) The exponents $\gamma_{\boldsymbol{\nu}}$ and, in principle, all amplitudes $f_{\boldsymbol{\nu},l}$ in the Puiseux expansion Eq. (4.4) of $g_{\boldsymbol{\nu}}(u_0)$ can be computed recursively from the functional equation (4.1). Compare the argument above for the case $\boldsymbol{\nu} = 0$. Our aim in this section is to determine the exponent $\gamma_{\boldsymbol{\nu}}$ and the amplitudes $f_{\boldsymbol{\nu},0}$. The method of Section 5 will also allow us to obtain information about the amplitudes $f_{\boldsymbol{\nu},l}$ for higher values of l, with reasonable effort.

We will first give an estimate of $\gamma_{\boldsymbol{\nu}}$.

Proposition 4.4. *Let Assumption 4.1 be satisfied. The exponent $\gamma_{\boldsymbol{\nu}}$ in the Puiseux expansion of $g_{\boldsymbol{\nu}}(u_0)$ Eqs. (4.3) and (4.4), satisfies the estimate*

$$\gamma_{\boldsymbol{\nu}} \le -\frac{1}{2} + \sum_{i=0}^{M} \left(1 + \frac{i}{2}\right) \nu_i. \tag{4.6}$$

Remark. Under mild additional assumptions, the above estimate is sharp, i.e., we have $\gamma_{\boldsymbol{\nu}} = -\frac{1}{2} + \sum_{i=0}^{M} \left(1 + \frac{i}{2}\right) \nu_i$. This follows from Proposition 4.5, where we show that the amplitudes $f_{\boldsymbol{\nu},0}$, for the special choice of $\gamma_{\boldsymbol{\nu}}$ as above, have non-zero values. Similar estimates can be obtained in situations different from a square-root singularity. The proof below can be adapted to treat these situations.

Proof. Set $\widetilde{\gamma}_{\boldsymbol{\nu}} = -r + \sum_{i=0}^{M} r_i \nu_i$, where $r = 1/2$, and $r_i = (1 + i/2)$ for $i = 0, \ldots, M$. We have $r_{k+1} > r_k$ for $k = 0, \ldots, M - 1$. For $\boldsymbol{\nu} = \mathbf{0}$ and $\boldsymbol{\nu} = \boldsymbol{e}_0$, we get from Proposition 4.2 that $\gamma_{\boldsymbol{\nu}} = \widetilde{\gamma}_{\boldsymbol{\nu}}$. We prove the proposition by induction on $\boldsymbol{\nu}$ w.r.t. the total order $\prec$ of Definition 2.2, using Proposition 4.3. For the remaining induction step, it suffices to show that $\gamma_{\boldsymbol{\nu}} \le \widetilde{\gamma}_{\boldsymbol{\nu}}$.

Assume that $\gamma_{\boldsymbol{\mu}} \le \widetilde{\gamma}_{\boldsymbol{\mu}}$ holds for $\boldsymbol{\mu} \prec \boldsymbol{\nu}$. We group all terms in Eq. (3.1), evaluated at $\boldsymbol{u} = \boldsymbol{u}_0$, which contain $g_{\boldsymbol{\nu}}(u_0)$, to the l.h.s. In the resulting equation, the l.h.s. $L(s)$ satisfies asymptotically, with c_L a non-zero constant,

$L(s) \sim c_L(u_c - u_0)^{-\gamma_{\boldsymbol{\nu}}+1/2}$ as $u_0 \to u_c^-$, see the remark following the proof of Proposition 4.3. By the reasoning in the proof of Proposition 4.3, the r.h.s. $R(s)$ of the equation satisfies asymptotically, with c_R a non-zero constant, $R(s) \sim c_R(u_c - u_0)^{-\gamma}$ as $u_0 \to u_c^-$. Using the induction assumption, the exponent γ can be estimated by

$$\gamma \le \sum_{i=1}^{s} \gamma_{\boldsymbol{\mu}_i} |\boldsymbol{\kappa}_i| \le \sum_{i=1}^{s} \left(-\frac{1}{2} + \sum_{j=0}^{M} r_j (\boldsymbol{\mu}_i)_j \right) |\boldsymbol{\kappa}_i| = -\frac{|\boldsymbol{\lambda}|}{2} + \sum_{j=0}^{M} r_j \nu_j = -\frac{(|\boldsymbol{\lambda}| - 1)}{2} + \widetilde{\gamma}_{\boldsymbol{\nu}}.$$

In the above expression, we used Lemma 2.3 and the properties of $p_s(\boldsymbol{\nu}, \boldsymbol{\lambda})$ in Lemma 3.2. Since $L(s)$ equals $R(s)$, we have $\gamma_{\boldsymbol{\nu}} - 1/2 = \gamma$. Thus, the estimate $\gamma_{\boldsymbol{\nu}} \le \widetilde{\gamma}_{\boldsymbol{\nu}}$ is satisfied for $|\boldsymbol{\lambda}| \ge 2$. Let us analyse the terms contributing to $|\boldsymbol{\lambda}| = 1$ in Lemma 3.2. We have $s = 1$, $|\boldsymbol{\kappa}_1| = 1$ and $\boldsymbol{\mu}_1 = \boldsymbol{\nu}$. Now, use Lemma 2.3. If in Eq. (2.1) we have $|\boldsymbol{\mu}| = |\boldsymbol{\nu}| - l$ for some $l \in \{1, \dots, |\boldsymbol{\nu}|\}$, it follows that $\widetilde{\gamma}_{\boldsymbol{\mu}} \le \widetilde{\gamma}_{\boldsymbol{\nu}} - l \min_{0 \le k \le M}\{r_k\} = \widetilde{\gamma}_{\boldsymbol{\nu}} - l$. Thus, terms with exponents larger than $\widetilde{\gamma}_{\boldsymbol{\nu}} - 1/2$ must have $l = 0$. If $|\boldsymbol{\mu}| = |\boldsymbol{\nu}|$ but $\boldsymbol{\mu} \ne \boldsymbol{\nu}$, we have $\widetilde{\gamma}_{\boldsymbol{\mu}} \le \widetilde{\gamma}_{\boldsymbol{\nu}} - |r_j - r_k|$ for some indices $j, k \in \{0, \dots, M\}$ satisfying $j \ne k$. This means that the exponent estimate $\gamma_{\boldsymbol{\nu}} - 1/2 \le -1/2 + \widetilde{\gamma}_{\boldsymbol{\nu}}$ is satisfied as long as $|r_j - r_k| \ge 1/2$ for $j, k \in \{0, \dots, M\}$ and $j \ne k$. This condition is satisfied for the particular choice of exponents $r_i = (1 + i/2)$ of the proposition. Thus, the proposition has been proved. □

The reasoning in the proof of Proposition 4.4 can be refined in order to obtain a recursion for the amplitudes $f_{\boldsymbol{\nu},0}$. For vectors $\boldsymbol{x} = (x_1, \dots, x_K) \in \mathbb{R}^K$ and $\boldsymbol{y} = (y_1, \dots, y_K) \in \mathbb{R}^K$, we write $\boldsymbol{x} \le \boldsymbol{y}$ if $x_i \le y_i$ for $i = 1, \dots, K$. In the proof, we will use the "large oh" symbol: For functions $f(x)$ and $h(x)$, we write $f(x) = \mathcal{O}(h(x))$ as $x \to x_0$ in a domain D, if there exist a positive constant C and a neighbourhood $N(x_0)$ of x_0 such that $|f(x)| \le C|h(x)|$ for all $x \in N(x_0) \cap D$.

Proposition 4.5. *Let Assumption 4.1 be satisfied and fix $\boldsymbol{\nu} \in \mathbb{N}_0^{1+M}$. Then, the amplitudes $f_{\boldsymbol{\nu},0} = f_{\boldsymbol{\nu}}$ of the Puiseux expansion Eq. (4.4) are, if $\boldsymbol{\nu} \ne \boldsymbol{0}$ and $\boldsymbol{\nu} \ne \boldsymbol{e}_0$, determined by the recursion*

$$f_{\boldsymbol{\nu}} = \mu_0 \gamma_{\boldsymbol{\nu}-\boldsymbol{e}_1} f_{\boldsymbol{\nu}-\boldsymbol{e}_1} + \sum_{i=1}^{M-1} \mu_i (\nu_i + 1) f_{\boldsymbol{\nu}-\boldsymbol{e}_{i+1}+\boldsymbol{e}_i} - \frac{1}{2 f_{\boldsymbol{0}}} \sum_{\substack{\boldsymbol{\rho} \ne \boldsymbol{0}, \boldsymbol{\rho} \ne \boldsymbol{\nu} \\ \boldsymbol{0} \le \boldsymbol{\rho} \le \boldsymbol{\nu}}} f_{\boldsymbol{\rho}} f_{\boldsymbol{\nu}-\boldsymbol{\rho}}, \tag{4.7}$$

with boundary conditions $f_{\boldsymbol{0}} = -\sqrt{C/B} < 0$, $f_{\boldsymbol{e}_0} = -f_{\boldsymbol{0}}/2 > 0$, and $f_{\boldsymbol{\nu}} = 0$ if $\nu_j < 0$ for some $j \in \{1, \dots, M\}$. We have

$$\gamma_{\boldsymbol{\nu}} = -\frac{1}{2} + \sum_{i=0}^{M} \left(1 + \frac{i}{2}\right) \nu_i.$$

For $i \in \{0, \dots, M-1\}$, we have $\mu_i = -A_i/(2B f_{\boldsymbol{0}}) \ge 0$, and the number $A_i \ge 0$ is given by

$$A_i = \sum_{j=1}^{N} \frac{\partial v_i^{(j)}}{\partial u_{i+1}} (\boldsymbol{u}_c) \frac{\partial P}{\partial y_j} (\boldsymbol{u}_c, \boldsymbol{G}(\boldsymbol{u}_c)), \tag{4.8}$$

where $\boldsymbol{u}_c = (u_c, 1, \dots, 1)$. If $\boldsymbol{\nu} \ne \boldsymbol{0}$, the amplitudes satisfy $f_{\boldsymbol{\nu}} \ge 0$. This inequality is strict, if $A_i > 0$ for $i = 0, \dots, M-1$.

Proof. The amplitude $f_{\boldsymbol{0}}$ has been determined in Proposition 4.2. Assume that $\boldsymbol{\nu} \ne \boldsymbol{0}$ and $\boldsymbol{\nu} \ne \boldsymbol{e}_0$. We group all terms in Eq. (3.1), evaluated at $\boldsymbol{u} = \boldsymbol{u}_0$, which contain $g_{\boldsymbol{\nu}}(u_0)$, to the l.h.s. Using Eqs. (3.2) and (2.9), the l.h.s. $L(\boldsymbol{u}_0)$ of the resulting equation is given by

$$L(\boldsymbol{u}_0) = \boldsymbol{\nu}! g_{\boldsymbol{\nu}}(u_0) P_{\boldsymbol{e}_1}(u_0, \boldsymbol{G}(\boldsymbol{u}_0)) / \left(\frac{\mathrm{d}G}{\mathrm{d}u_0} (\boldsymbol{u}_0) \right).$$

Here, we used the identity

$$\frac{\mathrm{d}G}{\mathrm{d}u_0}(\boldsymbol{u}_0) = \left(\sum_{j=1}^{N} \frac{\partial P}{\partial y_j} (\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0)) \right) \frac{\mathrm{d}G}{\mathrm{d}u_0}(\boldsymbol{u}_0) + P_{\boldsymbol{e}_1}(\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0)),$$

which is obtained by differentiating Eq. (4.1). By the reasoning in the proof of Proposition 4.4, the r.h.s. $R(u_0)$ of the resulting equation satisfies asymptotically, with c_R a non-zero constant, $R(u_0) \sim c_R(u_c - u_0)^{-(\gamma_{\boldsymbol{\nu}}-1/2)}$ as $u_0 \to u_c^-$.

We collect all terms with exponents $\gamma_{\boldsymbol{\nu}} - 1/2$. Due to the proof of Proposition 4.4, they arise from terms where $|\boldsymbol{\lambda}| = 1$ or $|\boldsymbol{\lambda}| = 2$ in Eq. (3.1). An explicit analysis using Lemma 2.3, whose details we omit, leads to the following expressions. The contribution arising from terms where $|\boldsymbol{\lambda}| = 2$ is given by

$$\frac{1}{2}\left(\sum_{j,k=1}^{N} \frac{\partial^2 P}{\partial y_j \partial y_k}(\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0))\right) \boldsymbol{\nu}! \sum_{\substack{\boldsymbol{\rho}\neq\boldsymbol{0},\boldsymbol{\rho}\neq\boldsymbol{\nu}\\ \boldsymbol{0}\le\boldsymbol{\rho}\le\boldsymbol{\nu}}} g_{\boldsymbol{\rho}}(u_0) g_{\boldsymbol{\nu}-\boldsymbol{\rho}}(u_0).$$

The contribution from terms where $|\boldsymbol{\lambda}| = 1$ is given by

$$\sum_{j=1}^{N} \frac{\partial P}{\partial y_j}(\boldsymbol{u}_0, \boldsymbol{G}(\boldsymbol{u}_0)) \sum_{i=0}^{M-1} (\boldsymbol{\nu} + \boldsymbol{e}_i)! \frac{\partial v_i^{(j)}}{\partial u_{i+1}}(\boldsymbol{u}_0) g_{\boldsymbol{\nu}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}(u_0).$$

Omitting arguments and normalising the l.h.s., we arrive at the equation

$$\begin{aligned} g_{\boldsymbol{\nu}} &= \frac{\sum \frac{\partial^2 P}{\partial y_j \partial y_k}}{2P_{\boldsymbol{e}_1}} G' \left(\sum g_{\boldsymbol{\rho}} g_{\boldsymbol{\nu}-\boldsymbol{\rho}}\right) + \frac{G'}{P_{\boldsymbol{e}_1}} \sum_{j=1}^{N} \frac{\partial P}{\partial y_j} \frac{\partial v_0^{(j)}}{\partial u_1} g'_{\boldsymbol{\nu}-\boldsymbol{e}_1} \\ &\quad + \frac{G'}{P_{\boldsymbol{e}_1}} \sum_{j=1}^{N} \frac{\partial P}{\partial y_j} \sum_{i=1}^{M} (\nu_i + 1) \frac{\partial v_i^{(j)}}{\partial u_{i+1}} g_{\boldsymbol{\nu}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}} + \mathcal{O}\left((u_c - u_0)^{-\gamma_{\boldsymbol{\nu}}+1/2}\right) \end{aligned}$$

as $u_0 \to u_c^-$, where $()'$ denotes differentiation w.r.t. u_0. We have $G' = g_{\boldsymbol{e}_1}$ and $f_{\boldsymbol{e}_1} = -f_{\boldsymbol{0}}/2 > 0$. The implied recursion for the amplitudes $f_{\boldsymbol{\nu},0}$ is given by

$$\begin{aligned} f_{\boldsymbol{\nu}} &= \frac{\sum \frac{\partial^2 P}{\partial y_j \partial y_k}}{2P_{\boldsymbol{e}_1}} \left(-\frac{f_{\boldsymbol{0}}}{2}\right) \left(\sum f_{\boldsymbol{\rho}} f_{\boldsymbol{\nu}-\boldsymbol{\rho}}\right) - \frac{f_{\boldsymbol{0}}}{2P_{\boldsymbol{e}_1}} \sum_{j=1}^{N} \frac{\partial P}{\partial y_j} \frac{\partial v_0^{(j)}}{\partial u_1} \gamma_{\boldsymbol{\nu}-\boldsymbol{e}_1} f_{\boldsymbol{\nu}-\boldsymbol{e}_1} \\ &\quad - \frac{f_{\boldsymbol{0}}}{2P_{\boldsymbol{e}_1}} \sum_{j=1}^{N} \frac{\partial P}{\partial y_j} \sum_{i=1}^{M} (\nu_i + 1) \frac{\partial v_i^{(j)}}{\partial u_{i+1}} f_{\boldsymbol{\nu}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}}. \end{aligned}$$

After rewriting prefactors, using Eq. (4.2) and Proposition 4.3, we arrive at Eq. (4.7).

We show strict positivity of the amplitudes, if $A_i > 0$ for $i = 0, \ldots, M - 1$. For $|\boldsymbol{\nu}| \ge 2$, all prefactors of $f_{\boldsymbol{\mu}}$ in Eq. (4.7) are non-negative. For the first term, this follows from the estimate

$$\gamma_{\boldsymbol{\nu}-\boldsymbol{e}_1} = \left(-\frac{1}{2} - \frac{3}{2} + \sum_{i=0}^{M} \frac{i+2}{2} \nu_i\right) \ge -2 + |\boldsymbol{\nu}| \ge 0.$$

Moreover, the last sum in Eq. (4.7) is over a non-empty set of indices $\boldsymbol{\rho}$ and has a strictly positive prefactor. For $|\boldsymbol{\nu}| = 1$ and $\boldsymbol{\nu} \neq \boldsymbol{e}_1$, the first term in Eq. (4.7) is zero, whereas the prefactors of $f_{\boldsymbol{\mu}}$ in the second term are strictly positive. Note finally that $f_{\boldsymbol{e}_1} = \mu_0 \gamma_{\boldsymbol{0}} f_{\boldsymbol{0}} > 0$, as is readily inferred from Eq. (4.7). This leads, by induction, to strictly positive amplitudes $f_{\boldsymbol{\nu}}$. The same argument shows that $f_{\boldsymbol{\nu}} \ge \boldsymbol{0}$ for $\boldsymbol{\nu} \neq \boldsymbol{0}$. □

5. Method of dominant balance

We now discuss an alternative method to obtain the recursion for the amplitudes $f_{\boldsymbol{\nu}}$ in Eq. (4.7), if $\nu_0 = 0$. It is based on a generating function approach, and generally easier to apply than the combinatorial approach in the proof of Proposition 4.5, which was based on an application of Faa di Bruno's formula. It also allows us to analyse corrections to the leading singular behaviour of the factorial moment generating functions. Introduce parameters $\delta_i = 1 - u_i$, where $i \in \{1, \ldots, M\}$, and set $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_M)$. From now on, we consider factorial moment generating functions $g_{\boldsymbol{\nu}}(u_0)$ for $\nu_0 = 0$ only. For $\boldsymbol{k} = (k_1, \ldots, k_M) \in \mathbb{N}_0^M$, set $g_{\boldsymbol{k}}(u_0) = g_{(0,\boldsymbol{k})}(u_0)$, $f_{\boldsymbol{k}}(u_0) = f_{(0,\boldsymbol{k})}(u_0)$, and $\gamma_{\boldsymbol{k}}(u_0) = \gamma_{(0,\boldsymbol{k})}(u_0)$.

Proposition 5.1. *Assume that the q-functional equation Eq. (2.5) has the solution $G(u_0, \boldsymbol{u}_+) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$ such that $G(0, \boldsymbol{u}_+) = 0$. Then $\widetilde{G}(u_0, \boldsymbol{\delta}) := G(u_0, 1-\delta_1, \ldots, 1-\delta_M) \in \mathbb{C}[[u_0, \boldsymbol{\delta}]]$ is a formal power series, given by*

$$\widetilde{G}(u_0, \boldsymbol{\delta}) = \sum_{\boldsymbol{k} \geq \boldsymbol{0}} (-1)^{\boldsymbol{k}} g_{\boldsymbol{k}}(u_0) \boldsymbol{\delta}^{\boldsymbol{k}},$$

where $g_{\boldsymbol{k}}(u_0) = g_{(0,\boldsymbol{k})}(u_0) \in \mathbb{C}[[u_0]]$ are the factorial moment generating functions Eq. (2.9).

Proof. Proposition 2.5 states that $G(\boldsymbol{u}) \in \mathcal{H}_{d(P)}(\boldsymbol{u}_+)[[u_0]]$. Thus $\widetilde{G}(u_0, \boldsymbol{\delta}) \in \mathcal{H}_{d(P)-1}(\boldsymbol{\delta})[[u_0]] \subset \mathbb{C}[[\boldsymbol{\delta}]][[u_0]] = \mathbb{C}[[u_0, \boldsymbol{\delta}]]$. We thus have

$$\widetilde{G}(u_0, \boldsymbol{\delta}) = \sum_{\boldsymbol{k} \geq \boldsymbol{0}} h_{\boldsymbol{k}}(u_0) \boldsymbol{\delta}^{\boldsymbol{k}},$$

for some $h_{\boldsymbol{k}}(u_0) \in \mathbb{C}[[u_0]]$. By Taylor's formula, we have

$$h_{\boldsymbol{k}}(u_0) = \frac{1}{\boldsymbol{k}!} \widetilde{G}_{(0,\boldsymbol{k})}(u_0, \boldsymbol{\delta}) \bigg|_{\boldsymbol{\delta}=\boldsymbol{0}} = \frac{(-1)^{\boldsymbol{k}}}{\boldsymbol{k}!} G_{(0,\boldsymbol{k})}(\boldsymbol{u}) \bigg|_{\boldsymbol{u}=\boldsymbol{u}_0} = (-1)^{\boldsymbol{k}} g_{\boldsymbol{k}}(u_0),$$

and the proposition is proved.

Let $\mathbb{C}((s))$ denote the field of formal Laurent series $f(s) = \sum_{l \geq l_0} f_l s^l$, where $l_0 \in \mathbb{Z}$ and $f_l \in \mathbb{C}$ for $l \geq l_0$. Employing the Puiseux expansions of the factorial moment generating functions, we have an alternative representation of the generating function $G(\boldsymbol{u})$.

Proposition 5.2. *Let Assumption 4.1 be satisfied. Replace the coefficients $g_{\boldsymbol{k}}(u_0)$ of $\widetilde{G}(u_0, \boldsymbol{\delta}) \in \mathbb{C}[[u_0, \boldsymbol{\delta}]]$ in Proposition 5.1 by their Puiseux expansions of Propositions 4.2 and 4.3, and denote the resulting series by $\widetilde{G}(s, \boldsymbol{\delta}) \in \mathbb{C}((s))[[\boldsymbol{\delta}]]$. It is explicitly given by*

$$\widetilde{G}(s, \boldsymbol{\delta}) = G(\boldsymbol{u}_c) + \sum_{\boldsymbol{k} \geq \boldsymbol{0}} \left(\sum_{l=0}^{\infty} (-1)^{\boldsymbol{k}} f_{\boldsymbol{k},l} s^{-\gamma_{\boldsymbol{k}} + l/2} \right) \boldsymbol{\delta}^{\boldsymbol{k}},$$

where $\gamma_{\boldsymbol{k}} = -1/2 + \sum_{i=1}^{M} (1 + i/2) k_i$, where the numbers $f_{\boldsymbol{k},l} = f_{(0,\boldsymbol{k}),l}$ are defined in Eq. (4.3) and in Eq. (4.4), and where $\boldsymbol{u}_c = (u_c, 1, \ldots, 1)$.

Then, the rescaled series $G(\boldsymbol{u}(s, \boldsymbol{\epsilon})) = \widetilde{G}(s, \epsilon_1 s^3, \epsilon_2 s^4, \ldots, \epsilon_M s^{M+2}) \in \mathbb{C}[[s, \boldsymbol{\epsilon}]]$ is a formal power series,

$$G(\boldsymbol{u}(s, \boldsymbol{\epsilon})) = G(\boldsymbol{u}_c) + s F(s, \boldsymbol{\epsilon}), \tag{5.1}$$

where $F(s, \boldsymbol{\epsilon}) \in \mathbb{C}[[s, \boldsymbol{\epsilon}]]$ is given by

$$F(s, \boldsymbol{\epsilon}) = \sum_{l=0}^{\infty} F_l(\boldsymbol{\epsilon}) s^l, \qquad F_l(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{k} \geq \boldsymbol{0}} (-1)^{\boldsymbol{k}} f_{\boldsymbol{k},l} \boldsymbol{\epsilon}^{\boldsymbol{k}}. \tag{5.2}$$

Proof. Due to Propositions 4.2 and 4.3, we have $g_{\boldsymbol{k}}(u_0) \in \mathbb{C}((s))$, where $s = \sqrt{u_c - u_0}$. The explicit form Eq. (5.1) follows immediately from the Puiseux expansions in Propositions 4.2 and 4.3, together with the exponent estimate in Proposition 4.4. $\square$

Remark. (i) Equations like Eq. (5.1) appear in statistical mechanics as a so-called *scaling Ansatz*, being an assumption on the behaviour of a generating function near a multicritical point singularity [26], see also [12,32]. Its validity has been proved only in a limited number of examples. Here, we employ the different framework of formal power series.

(ii) Proposition 5.2 suggests an alternative strategy to compute the singular behaviour of the factorial moment generating functions. We consider the functional equation for $F(s, \boldsymbol{\epsilon}) \in \mathbb{C}[[s, \boldsymbol{\epsilon}]]$, which is induced by the q-functional equation for $G(\boldsymbol{u})$. Writing $F(s, \boldsymbol{\epsilon}) = \sum_{l \geq 0} F_l(\boldsymbol{\epsilon}) s^l$, where $F_l(\boldsymbol{\epsilon}) \in \mathbb{C}[[\boldsymbol{\epsilon}]]$, then leads to a partial differential equation for $F_l(\boldsymbol{\epsilon})$, upon expanding the induced functional equation in powers of s. Note that this algorithm, which is computationally involved, can be easily implemented in a computer algebra system. In statistical mechanics, the above method is called the method of *dominant balance*, see [44,46]. It can be applied in our framework, if an exponent bound like that of Proposition 4.4 is known. Such bounds are generally easier to obtain than explicit recursions for amplitudes, see the above proofs.

Proposition 5.3. *Let Assumption 4.1 be satisfied. Define the power series $\boldsymbol{u}(s,\boldsymbol{\epsilon}) = (u_0(s,\boldsymbol{\epsilon}),\ldots,u_M(s,\boldsymbol{\epsilon}))$, where*

$$u_0(s,\boldsymbol{\epsilon}) = u_c - s^2, \qquad u_i(s,\boldsymbol{\epsilon}) = 1 - \epsilon_i s^{i+2} \qquad (i = 1,\ldots,M).$$

For a q-shift $\boldsymbol{v}$, define the induced q-shift $s_{\boldsymbol{v}}$ of s and the induced q-shift $\boldsymbol{\epsilon}_{\boldsymbol{v}} = (\epsilon_{1,\boldsymbol{v}},\ldots,\epsilon_{M,\boldsymbol{v}})$ of $\boldsymbol{\epsilon}$ by

$$s_{\boldsymbol{v}} = s_{\boldsymbol{v}}(s,\boldsymbol{\epsilon}) = \sqrt{u_c - v_0(\boldsymbol{u}(s,\boldsymbol{\epsilon}))},\, \epsilon_{i,\boldsymbol{v}} = \epsilon_{i,\boldsymbol{v}}(s,\boldsymbol{\epsilon}) = \frac{1 - v_i(\boldsymbol{u}(s,\boldsymbol{\epsilon}))}{s_{\boldsymbol{v}}(s,\boldsymbol{\epsilon})^{i+2}} \quad (i = 1,\ldots,M).$$

We then have $s_{\boldsymbol{v}} \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$ and $\epsilon_{i,\boldsymbol{v}} \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$ for $i = 1,\ldots,M$. The functional equation for $F(s,\boldsymbol{\epsilon}) \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$ of Proposition 5.2, induced by Eq. (2.5), is given by

$$G(\boldsymbol{u}(s,\boldsymbol{\epsilon})) = P(\boldsymbol{u}(s,\boldsymbol{\epsilon}), \boldsymbol{H}(\boldsymbol{u}(s,\boldsymbol{\epsilon}))). \tag{5.3}$$

In the above equation, $G(\boldsymbol{u}(s,\boldsymbol{\epsilon}))$ is given by Eq. (5.1), and the power series $H^{(j)}(\boldsymbol{u}(s,\boldsymbol{\epsilon}))$ are given by

$$H^{(j)}(\boldsymbol{u}(s,\boldsymbol{\epsilon})) = G(\boldsymbol{u}_c) + s_{\boldsymbol{v}^{(j)}} F(s_{\boldsymbol{v}^{(j)}}, \boldsymbol{\epsilon}_{\boldsymbol{v}^{(j)}}) \quad (1 \le j \le N).$$

Proof. This follows from direct computation. Note first that the Taylor expansions of the power series $v_k(\boldsymbol{u})$ about $\boldsymbol{u} = \boldsymbol{u}_0$ are of the form

$$v_k(\boldsymbol{u}) = v_k(\boldsymbol{u}_0) + \sum_{\substack{\boldsymbol{l}\neq 0\\ \boldsymbol{l}\geq 0}} \frac{(-1)^{\boldsymbol{l}}}{\boldsymbol{l}!} v_{k,\boldsymbol{l}}(\boldsymbol{u}_0)(1-u_1)^{l_1}\cdot\cdots\cdot(1-u_M)^{l_M} \qquad (k = 0,\ldots,M). \tag{5.4}$$

Now insert the parametrisations $u_k(s,\boldsymbol{\epsilon}) \in \mathbb{C}[s,\boldsymbol{\epsilon}]$, where $k = 0,\ldots,M$. We get $v_k(\boldsymbol{u}(s,\boldsymbol{\epsilon})) \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$. Due to the q-shift properties, we have

$$\begin{aligned}
v_0(\boldsymbol{u}(s,\boldsymbol{\epsilon})) &= u_c - s^2 - \frac{\partial v_0}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1 s^3 + \mathcal{O}(s^4),\\
v_k(\boldsymbol{u}(s,\boldsymbol{\epsilon})) &= 1 - \epsilon_k s^{k+2} - \frac{\partial v_k}{\partial u_{k+1}}(\boldsymbol{u}_c)\epsilon_{k+1} s^{k+3} + \mathcal{O}(s^{k+4}) \quad (k = 1,\ldots,M-1),\\
v_M(\boldsymbol{u}(s,\boldsymbol{\epsilon})) &= 1 - \epsilon_M s^{M+2} + \mathcal{O}(s^{M+3}).
\end{aligned}$$

We thus have $v_0(\boldsymbol{u}(s,\boldsymbol{\epsilon})) = u_c - s^2 - s^3 R_0(s,\boldsymbol{\epsilon})$ and $v_k(\boldsymbol{u}(s,\boldsymbol{\epsilon})) = 1 - s^{k+2} R_k(s,\boldsymbol{\epsilon})$ for $k = 1,\ldots,M$, where $R_i(s,\boldsymbol{\epsilon}) \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$ for $i = 0,\ldots,M$. For $s_{\boldsymbol{v}}$, we get

$$s_{\boldsymbol{v}} = \sqrt{u_c - v_0(\boldsymbol{u}(s,\boldsymbol{\epsilon}))} = s\sqrt{1 - sR_0(s,\boldsymbol{\epsilon})}.$$

We thus have $s_{\boldsymbol{v}} \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$ and $s_{\boldsymbol{v}}(0,\boldsymbol{\epsilon}) = 0$. We get

$$\epsilon_{i,\boldsymbol{v}} = \frac{1 - v_i(\boldsymbol{u}(s,\boldsymbol{\epsilon}))}{s_{\boldsymbol{v}}^{i+2}} = \frac{R_i(s,\boldsymbol{\epsilon})}{\sqrt{1 - sR(s,\boldsymbol{\epsilon})}^{\,i+2}} \quad (i = 1,\ldots,M).$$

It follows that $\epsilon_{i,\boldsymbol{v}} \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$, where $i = 1,\ldots,M$. Furthermore, we get from Eq. (5.4)

$$\epsilon_{i,\boldsymbol{v}}(0,\boldsymbol{\epsilon}) = R_i(0,\boldsymbol{\epsilon}) = \epsilon_i,$$

such that $\boldsymbol{\epsilon}_{\boldsymbol{v}}(0,\boldsymbol{\epsilon}) = \boldsymbol{\epsilon}$. This ensures that Eq. (5.3) is well defined for $F(s,\boldsymbol{\epsilon}) \in \mathbb{C}[[s,\boldsymbol{\epsilon}]]$. □

Proposition 5.4. *Let Assumption 4.1 be satisfied. The formal power series $F_0(\boldsymbol{\epsilon}) \in \mathbb{C}[[\boldsymbol{\epsilon}]]$ satisfies the singular, quasi-linear partial differential equation of first order*

$$\sum_{j=1}^{N} \frac{\partial P}{\partial y_j}(\boldsymbol{u}_c, \boldsymbol{G}(\boldsymbol{u}_c)) \left(\frac{1}{2}\frac{\partial v_0^{(j)}}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1 F_0(\boldsymbol{\epsilon}) + \sum_{i=1}^{M} h_i^{(j)}(\boldsymbol{\epsilon}) \frac{\partial}{\partial \epsilon_i} F_0(\boldsymbol{\epsilon}) \right) + BF_0^2(\boldsymbol{\epsilon}) - C = 0, \tag{5.5}$$

where the numbers B, C are defined in Eq. (4.2), and the formal power series $h_i^{(j)}(\boldsymbol{\epsilon})$ are given by

$$\begin{aligned} h_i^{(j)}(\boldsymbol{\epsilon}) &= \frac{\partial v_i^{(j)}}{\partial u_{i+1}}(\boldsymbol{u}_c)\epsilon_{i+1} - \frac{i+2}{2}\frac{\partial v_0^{(j)}}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1\epsilon_i \quad (i = 1, \dots, M-1), \\ h_M^{(j)}(\boldsymbol{\epsilon}) &= -\frac{M+2}{2}\frac{\partial v_0^{(j)}}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1\epsilon_M. \end{aligned} \tag{5.6}$$

Proof. Using the expansions in the proof of Proposition 5.3, it is readily inferred that the power series $s_{\boldsymbol{v}}$ and $\boldsymbol{\epsilon}_{\boldsymbol{v}}$ are, to leading orders in s, given by

$$\begin{aligned} s_{\boldsymbol{v}^{(j)}} &= s + \frac{1}{2}\frac{\partial v_0^{(j)}}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1 s^2 + \mathcal{O}(s^3), \\ \epsilon_{i,\boldsymbol{v}^{(j)}} &= \epsilon_i + h_i^{(j)}(\boldsymbol{\epsilon})s + \mathcal{O}(s^2) \quad (i = 1, \dots, M-1), \end{aligned}$$

with $h_i^{(j)}(\boldsymbol{\epsilon})$ as defined in Eq. (5.6). Using this result, we compute the expansion of $H^{(j)}(\boldsymbol{u}(s,\boldsymbol{\epsilon}))$ up to order s^2. This yields

$$\begin{aligned} H^{(j)}(\boldsymbol{u}(s,\boldsymbol{\epsilon})) &= G(\boldsymbol{u}_c) + s_{\boldsymbol{v}^{(j)}} F(s_{\boldsymbol{v}^{(j)}}, \boldsymbol{\epsilon}_{\boldsymbol{v}^{(j)}}) \\ &= G(\boldsymbol{u}_c) + F_0(\boldsymbol{\epsilon})s + \left(F_1(\boldsymbol{\epsilon}) + \frac{1}{2}\frac{\partial v_0^{(j)}}{\partial u_1}(\boldsymbol{u}_c)\epsilon_1 F_0(\boldsymbol{\epsilon}) + \sum_{i=1}^{M} h_i^{(j)}(\boldsymbol{\epsilon})\frac{\partial}{\partial \epsilon_i}F_0(\boldsymbol{\epsilon}) \right) s^2 + \mathcal{O}(s^3). \end{aligned}$$

Now expand the functional equation (5.3) to leading orders in s. Terms of order s^0 vanish due to Eq. (4.1), evaluated at $\boldsymbol{u}_0 = \boldsymbol{u}_c$. Terms of order s^1 vanish due to the condition $\sum_{j=1}^{N}\frac{\partial P}{\partial y_j}(\boldsymbol{u}_c, \boldsymbol{G}(\boldsymbol{u}_c)) = 1$. Terms of order s^2 lead to the partial differential equation given above. □

Remark. (i) For $M > 2$, it is an open question whether closed form solutions for $F_0(\boldsymbol{\epsilon})$ exist. See [40] for a discussion of the cases $M = 1$ and $M = 2$.

(ii) The above method can also be used to derive partial differential equations characterising the generating functions $F_l(s,\boldsymbol{\epsilon})$ of the amplitudes $f_{\boldsymbol{k},l}$ for $l > 0$. These equations arise in the expansion of the q-functional equation in s at order $l+2$, see [46] for examples where $M = 1$.

The above theorem leads to an alternative derivation of the recursion Eq. (4.7) of Proposition 4.5 in the case $\nu_0 = 0$.

Proof (*Alternative Proof of Eq.* (4.7)). We set $F_0(\boldsymbol{\epsilon}) = K(-\boldsymbol{\epsilon}) + f_{\boldsymbol{0},0}$ and rewrite Eq. (5.5) in the form

$$K(\boldsymbol{\epsilon}) = \sum_{i=1}^{M}\frac{i+2}{2}\mu_0\epsilon_1\epsilon_i\frac{\partial K(\boldsymbol{\epsilon})}{\partial \epsilon_i} + \sum_{i=1}^{M-1}\mu_i\epsilon_{i+1}\frac{\partial K(\boldsymbol{\epsilon})}{\partial \epsilon_i} - \frac{\mu_0}{2}\epsilon_1(K(\boldsymbol{\epsilon}) + f_{\boldsymbol{0},0}) - \frac{1}{2f_{\boldsymbol{0},0}}K(\boldsymbol{\epsilon})^2, \tag{5.7}$$

where $K(\boldsymbol{0}) = 0$, and the constants μ_i are, for $i \in \{0, \dots, M-1\}$, given by $\mu_i = -A_i/(2Bf_{\boldsymbol{0},0})$, with A_i defined in Eq. (4.8). This leads to the recursion Eq. (4.7) for the coefficients $f_{\boldsymbol{\nu}}$ in Proposition 4.5, if we set $\boldsymbol{\nu} = (0, \boldsymbol{k})$. □

6. Growth of amplitudes

We are interested in the growth of the coefficients $f_{\boldsymbol{\nu},0}$, which appear in Proposition 4.5 and in Proposition 5.4, in the case $\nu_0 = 0$. To this end, we study properties of the partial differential equation of the associated generating function $F_0(\boldsymbol{\epsilon})$ of Eq. (5.2), given by

$$F_0(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{k}\geq\boldsymbol{0}}(-1)^{\boldsymbol{k}} f_{\boldsymbol{k},0}\boldsymbol{\epsilon}^{\boldsymbol{k}}.$$

Proposition 6.1. *For a q-functional equation, let Assumption 4.1 be satisfied. There exist positive real numbers $D, R_1, \dots, R_M$, such that*

$$|f_{\boldsymbol{k},0}| \leq D(k_1 + \dots + k_M)!\,(R_1)^{k_1} \cdot \dots \cdot (R_M)^{k_M}$$

for all $\boldsymbol{k} \geq \boldsymbol{0}$.

For the proof of the proposition, we apply the technique of majorising series. A formal power series $g = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$ majorises a formal power series $h = \sum_{\boldsymbol{k}} h_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$ if $|g_{\boldsymbol{k}}| \leq |\boldsymbol{k}|! h_{\boldsymbol{k}}$ for all $\boldsymbol{k} \geq \boldsymbol{0}$. We then write $g \ll h$. We have the following relations.

Lemma 6.2. *Let* $g \ll h$. *Then*

$$g^2 \ll h^2, \qquad x_1 g \ll x_1 h, \qquad x_1 x_i \frac{\partial g}{\partial x_i} \ll x_1 h, \qquad x_{j+1} \frac{\partial g}{\partial x_j} \ll x_{j+1} \frac{\partial h}{\partial x_j}, \tag{6.1}$$

for $i = 1, \ldots, M$ *and* $j = 1, \ldots, M-1$.

Proof (*Sketch of Proof*). These relations are checked by direct computation. If $i = 2, \ldots, M$, we have $x_1 x_i \frac{\partial g}{\partial x_i} = \sum k_i g_{\boldsymbol{k}-\boldsymbol{e}_1} \boldsymbol{x}^{\boldsymbol{k}}$. Thus $|k_i g_{\boldsymbol{k}-\boldsymbol{e}_1}| \leq (|\boldsymbol{k}|-1)! k_i h_{\boldsymbol{k}-\boldsymbol{e}_1} \leq |\boldsymbol{k}|! h_{\boldsymbol{k}-\boldsymbol{e}_1}$, and the statement follows. The remaining assertions are proved similarly. □

Proof (*Proof of* Proposition 6.1). Starting with $K(\boldsymbol{\epsilon})$, defined in Eq. (5.7), we introduce the majorant equation

$$L(\boldsymbol{\epsilon}) = \mu_0 \left(\sum_{i=1}^{M} \frac{i+2}{2} \right) \epsilon_1 L(\boldsymbol{\epsilon}) + \frac{\mu_0}{2} \epsilon_1 (L(\boldsymbol{\epsilon}) + |f_{\boldsymbol{0},0}|) + \frac{1}{2|f_{\boldsymbol{0},0}|} L(\boldsymbol{\epsilon})^2 + \sum_{i=1}^{M-1} \mu_i \epsilon_{i+1} \frac{\partial L(\boldsymbol{\epsilon})}{\partial \epsilon_i}. \tag{6.2}$$

The last equation uniquely defines a power series with non-negative coefficients and positive radius of convergence, satisfying $L(\boldsymbol{0}) = 0$. It belongs to a class of singular partial differential equations with regular solutions. It is discussed in [25, Thm. 2.8.2.1 and Sec. 2.9.5]. To prove that $K(\boldsymbol{\epsilon}) \ll L(\boldsymbol{\epsilon})$, denote the r.h.s. of Eq. (5.7) by RK and the r.h.s. of Eq. (6.2) by $\widehat{R}L$. By construction, $g \ll h$ implies $Rg \ll \widehat{R}h$. We use an iteration argument. Set $K_0 = L_0 = 0$. Clearly $K_0 \ll L_0$. Define $K_n = RK_{n-1}$ and $L_n = \widehat{R}L_{n-1}$ for $n \in \mathbb{N}$. We have $K_n \ll L_n$ for $n \in \mathbb{N}_0$, due to Lemma 6.2. Thus $K(\boldsymbol{\epsilon}) \ll L(\boldsymbol{\epsilon})$ for the formal solutions $K(\boldsymbol{\epsilon})$ of Eq. (5.7) and $L(\boldsymbol{\epsilon})$ of Eq. (6.2). Since $L(\boldsymbol{\epsilon})$ is regular, we have the estimate

$$|f_{\boldsymbol{k},0}| \leq |\boldsymbol{k}|! [\boldsymbol{\epsilon}^{\boldsymbol{k}}] L(\boldsymbol{\epsilon}) \leq D |\boldsymbol{k}|! \boldsymbol{R}^{\boldsymbol{k}}$$

for some real constants $D > 0$ and $R_i > 0$, where $i = 1, \ldots, M$. □

7. Moments and limit distributions

In the following, we give a probabilistic interpretation of the obtained results. This generalises the discussion of Dyck paths in the introduction, before Proposition 1.4, to the case of a general q-functional equation, and will prove part (i) of Theorem 1.5. For technical reasons (Lemma 7.4), we will not use random variables below, but rather argue using the associated probability measures.

For a q-functional equation, let Assumption 4.1 be satisfied. Then, the coefficients $p_{\boldsymbol{n}}$ in a solution $G(\boldsymbol{u}) = \sum_{\boldsymbol{n} \geq 0} p_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$, such that $G(\boldsymbol{0}) = 0$, are non-negative. Assume that the numbers A_i of Proposition 4.5 satisfy $A_i > 0$ for $i = 0, \ldots, M-1$. We then have $f_{\boldsymbol{k},0} > 0$ for all $\boldsymbol{k} \geq \boldsymbol{0}$ and $\boldsymbol{k} \neq \boldsymbol{0}$. This implies that $\sum_{n_1,\ldots,n_M} p_{n_0,n_1,\ldots,n_M} > 0$ for almost all n_0, see Eq. (7.2) below. Fix $N_0 \in \mathbb{N}$ such that strict positivity holds for all $n_0 \geq N_0$. For $n_0 \geq N_0$, we define discrete Borel probability measures $\widetilde{\mu}_{n_0}$ by

$$\widetilde{\mu}_{n_0} = \sum_{n_1,\ldots,n_M} \frac{p_{n_0,n_1,\ldots,n_M}}{\sum\limits_{m_1,\ldots,m_M} p_{n_0,m_1,\ldots,m_M}} \delta_{(n_1,\ldots,n_M)},$$

(compare Eq. (1.13) in the introduction). Their corresponding moments are, for $\boldsymbol{k} \in \mathbb{N}_0^M$, given by

$$\widetilde{m}_{\boldsymbol{k}}(n_0) = \frac{\sum\limits_{n_1,\ldots,n_M} n_1^{k_1} \cdot \ldots \cdot n_M^{k_M} p_{n_0,n_1,\ldots,n_M}}{\sum\limits_{n_1,\ldots,n_M} p_{n_0,n_1,\ldots,n_M}}. \tag{7.1}$$

We are interested in the asymptotic behaviour of the moments $\widetilde{m}_{\boldsymbol{k}}(n_0)$, as n_0 tends to infinity. This will be obtained by an analysis of the coefficients of the factorial moment generating functions, which relies on a transfer lemma [22, Thm. 1].

Lemma 7.1 (*Transfer Lemma [22]*). *Suppose that $F(z) = \sum_{n\geq 0} f_n z^n$ has a singularity at $z = z_c$ and is Δ-regular, i.e., it is analytic in the domain*

$$\Delta = \{z : |z| \leq z_c + \eta, |\arg(z - z_c)| \geq \phi\}$$

for some $\eta > 0$ and $0 < \phi < \pi/2$. Assume that, as $z \to z_c$ in Δ,

$$F(z) = \mathcal{O}\left((z_c - z)^{\alpha}\right)$$

for some real α. Then, the nth Taylor coefficient f_n of $F(z)$ satisfies

$$f_n = \mathcal{O}(z_c^{-n} n^{-1-\alpha}) \qquad (n \to \infty). \quad \square$$

The following lemma characterises the asymptotic behaviour of the moments $\widetilde{m}_{\boldsymbol{k}}(n_0)$, as n_0 tends to infinity.

Lemma 7.2. *Let Assumption 4.1 be satisfied. Assume that the numbers A_i of Proposition 4.5 satisfy $A_i > 0$ for $i = 0, \ldots, M - 1$. Then the moments $\widetilde{m}_{\boldsymbol{k}}(n_0)$ Eq. (7.1) are well defined for almost all n_0. They are for $\boldsymbol{k} \in \mathbb{N}_0^M$ asymptotically given by*

$$\widetilde{m}_{\boldsymbol{k}}(n_0) = \frac{\boldsymbol{k}!}{f_{\boldsymbol{0},0} u_c^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}}} \frac{\Gamma(\gamma_{\boldsymbol{0}})}{\Gamma(\gamma_{\boldsymbol{k}})} f_{\boldsymbol{k},0} n_0^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}} + \mathcal{O}(n_0^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}-1/2}) \quad (n_0 \to \infty),$$

where $\Gamma(z)$ denotes the Gamma function, and where the numbers $f_{\boldsymbol{k},0}$ and $\gamma_{\boldsymbol{k}}$ are defined in Proposition 4.5.

Proof. The functions $g_{\boldsymbol{k}}(u_0)$ are Δ-regular due to Proposition 4.3. We infer from Eq. (4.4) that $g_{\boldsymbol{k}}(u_0) = f_{\boldsymbol{k},0}(u_c - u_0)^{-\gamma_{\boldsymbol{k}}} + \mathcal{O}((u_c - u_0)^{-\gamma_{\boldsymbol{k}}+1/2})$ as $u_0 \to u_c^-$, where $\gamma_{\boldsymbol{0}} = -1/2$ and $\gamma_{\boldsymbol{k}} > 0$ otherwise. The coefficient asymptotics of $(u_c - u_0)^{-\gamma_{\boldsymbol{k}}}$ and the transfer lemma yield

$$[u_0^{n_0}] g_{\boldsymbol{k}}(u_0) = \frac{f_{\boldsymbol{k},0}}{u_c^{\gamma_{\boldsymbol{k}}} \Gamma(\gamma_{\boldsymbol{k}})} u_c^{-n_0} n_0^{\gamma_{\boldsymbol{k}}-1} \left(1 + \mathcal{O}(n_0^{-1/2})\right) \quad (n_0 \to \infty), \tag{7.2}$$

where $[x^n]f(x)$ denotes the coefficient of x^n in the power series $f(x)$. The error term implies that asymptotically factorial moments coincide with ordinary moments. The numbers $\widetilde{m}_{\boldsymbol{k}}(n_0)$ are thus well defined for n_0 large enough, and asymptotically given by

$$\begin{aligned}
\widetilde{m}_{\boldsymbol{k}}(n_0) &= \frac{\sum\limits_{\boldsymbol{n}_+\geq 0} \boldsymbol{n}_+^{\boldsymbol{k}} p_{n_0,\boldsymbol{n}_+}}{\sum\limits_{\boldsymbol{n}_+\geq 0} p_{n_0,\boldsymbol{n}_+}} = \frac{[u_0^{n_0}] g_{\boldsymbol{k}}(u_0)}{[u_0^{n_0}] g_{\boldsymbol{0}}(u_0)} \boldsymbol{k}! \left(1 + \mathcal{O}(n_0^{-1/2})\right) \\
&= \frac{\boldsymbol{k}!}{f_{\boldsymbol{0},0} u_c^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}}} \frac{\Gamma(\gamma_{\boldsymbol{0}})}{\Gamma(\gamma_{\boldsymbol{k}})} f_{\boldsymbol{k},0} n_0^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}} \left(1 + \mathcal{O}(n_0^{-1/2})\right) \quad (n_0 \to \infty).
\end{aligned}$$

This concludes the proof of the lemma. $\square$

The moments $\widetilde{m}_{\boldsymbol{k}}(n_0)$ diverge as $n_0 \to \infty$, as may be inferred from Lemma 7.2. Introduce normalised Borel probability measures μ_{n_0} by

$$\mu_{n_0} = \sum_{n_1,\ldots,n_M} \frac{p_{n_0,n_1,\ldots,n_M}}{\sum\limits_{m_1,\ldots,m_M} p_{n_0,m_1,\ldots,m_M}} \delta_{(\widetilde{n}_1,\ldots,\widetilde{n}_M)}, \tag{7.3}$$

where $\widetilde{n}_k = n_k n_0^{-(k+2)/2}$ for $k = 1, \ldots, M$. For $\boldsymbol{k} \in \mathbb{N}_0^M$, denote the corresponding moments by $m_{\boldsymbol{k}}(n_0)$. We now show that the limits

$$m_{\boldsymbol{k}} = \lim_{n_0\to\infty} m_{\boldsymbol{k}}(n_0) = \lim_{n_0\to\infty} \frac{\widetilde{m}_{\boldsymbol{k}}(n_0)}{n_0^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}}} = \frac{\boldsymbol{k}!}{f_{\boldsymbol{0},0} u_c^{\gamma_{\boldsymbol{k}}-\gamma_{\boldsymbol{0}}}} \frac{\Gamma(\gamma_{\boldsymbol{0}})}{\Gamma(\gamma_{\boldsymbol{k}})} f_{\boldsymbol{k},0} \tag{7.4}$$

exist and define a unique Borel probability measure μ, with finite moments $m_{\boldsymbol{k}} > 0$ at all orders. This will be achieved using Lévy's continuity theorem. We first prove the following lemma.

Lemma 7.3. *Let Assumption 4.1 be satisfied. Assume that the numbers A_i of Proposition 4.5 satisfy $A_i > 0$ for $i = 0, \dots, M-1$. Consider for $\boldsymbol{k} \in \mathbb{N}_0^M$ the numbers $m_{\boldsymbol{k}} \geq 0$, defined in Eq. (7.4) and Eq. (7.1). For $\boldsymbol{t} \in \mathbb{R}^M$, we have*

$$\lim_{|\boldsymbol{k}| \to \infty} \frac{m_{\boldsymbol{k}} \boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!} = 0.$$

Equivalently, the exponential generating function of the numbers $m_{\boldsymbol{k}}$ is entire.

Proof. The limit $m_{\boldsymbol{k}}$ Eq. (7.4) exists for $\boldsymbol{k} \in \mathbb{N}_0^M$, due to Lemma 7.2. Note that

$$\gamma_{\boldsymbol{k}} = -\frac{1}{2} + \sum_{i=1}^{M} \left(1 + \frac{i}{2}\right) k_i \geq -\frac{1}{2} + \frac{3}{2}|\boldsymbol{k}| \geq |\boldsymbol{k}| + 1 + \left\lfloor \frac{|\boldsymbol{k}| - 3}{2} \right\rfloor.$$

Thus $|\boldsymbol{k}| \to \infty$ implies $\gamma_{\boldsymbol{k}} \to \infty$. Furthermore, since $e(n/e)^n \leq n! \leq en(n/e)^n$ for $n \in \mathbb{N}$, we have the estimate

$$\frac{n!}{(n+n_0)!} \leq \frac{n(n/e)^n}{((n+n_0)/e)^{n+n_0}} \leq \frac{e^{n_0}}{n^{n_0-1}}$$

for $n, n_0 \in \mathbb{N}$. Now fix $\boldsymbol{t} \in \mathbb{R}^M$. Using Proposition 6.1, we estimate

$$\begin{aligned} \left| \frac{m_{\boldsymbol{k}} \boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!} \right| &= \frac{\Gamma(\gamma_0)}{\Gamma(\gamma_{\boldsymbol{k}})} \frac{|f_{\boldsymbol{k},0}|}{f_{\boldsymbol{0},0} u_c^{\gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{0}}}} |\boldsymbol{t}^{\boldsymbol{k}}| \leq \frac{\Gamma(\gamma_0)}{(|\boldsymbol{k}| + \lfloor \frac{|\boldsymbol{k}|-3}{2} \rfloor)!} \frac{D|\boldsymbol{k}|! \boldsymbol{R}^{\boldsymbol{k}}}{f_{\boldsymbol{0},0}} |\boldsymbol{t}^{\boldsymbol{k}}| \max(u_c, u_c^{-1})^{\gamma_{\boldsymbol{k}} - \gamma_{\boldsymbol{0}}} \\ &\leq \frac{D\Gamma(\gamma_0)}{f_{\boldsymbol{0},0}} e^{|\boldsymbol{k}|} \boldsymbol{R}^{\boldsymbol{k}} |\boldsymbol{t}^{\boldsymbol{k}}| \frac{1}{|\boldsymbol{k}|^{\lfloor \frac{|\boldsymbol{k}|-5}{2} \rfloor}} \max(u_c, u_c^{-1})^{\frac{M+2}{2}|\boldsymbol{k}|}. \end{aligned}$$

The r.h.s. vanishes as $|\boldsymbol{k}| \to \infty$, which implies the assertion. The equivalent statement is now obvious. □

Our proof of claim (i) in Theorem 1.5 relies on an application of Lévy's continuity theorem [5, Thm. 23.8], which we cite in the following lemma.

Lemma 7.4 (*Lévy's Continuity Theorem [5]*). *For $n \in \mathbb{N}$, let probability measures μ_n on the Borel σ-algebra of $\mathbb{R}^M$ be given. If the sequence $\{\widehat{\mu}_n\}_{n \in \mathbb{N}}$ of their characteristic functions $\widehat{\mu}_n$ converges pointwise to a complex function ϕ which is continuous at the origin, then ϕ is the characteristic function of a uniquely determined Borel probability measure μ. Moreover, $\{\mu_n\}_{n \in \mathbb{N}}$ converges to μ weakly.* □

Proof (*Proof of Claim (i) in Theorem 1.5.*). Lemma 7.2 implies that the Borel probability measures μ_{n_0} Eq. (7.3) are well defined for almost all $n_0 \in \mathbb{N}$. The estimate in Lemma 7.3 implies uniform convergence of the sequence of Fourier transforms $\widehat{\mu}_{n_0} : \mathbb{R}^M \to \mathbb{C}$ of μ_{n_0}, in every ball of finite radius centred at the origin. In particular, we have pointwise convergence of the sequence $\{\widehat{\mu}_{n_0}\}_{n_0 \in \mathbb{N}}$. For $M = 1$, the corresponding argument is given in the proof of [14, Thm 6.4.5]. It can be directly extended to arbitrary M. Since the functions $\widehat{\mu}_{n_0}$ are continuous, we conclude that the limit function $\phi : \mathbb{R}^M \to \mathbb{C}$ is continuous at the origin. Now apply Lévy's continuity theorem. The limit probability measure μ has moments $m_{\boldsymbol{k}}$ Eq. (7.4). The claimed statement of the theorem follows, when phrasing the result in terms of the associated random variables. □

The connection to Brownian motion, which is claimed in part (ii) of Theorem 1.5, will be established in the following section.

8. Dyck paths revisited

We apply our results to the example of Dyck paths of Section 1.2. The power series solution $E(\boldsymbol{u}_0)$ of Eq. (2.6), specialised to $\boldsymbol{u} = \boldsymbol{u}_0$, has radius of convergence $u_c = 1/4$, with a square-root singularity at $u = u_c$, and $E(\boldsymbol{u}_c) = 1$. The q-functional equation (2.6) satisfies Assumption 4.1. Since the random variables $(X_{1,n_0}, \dots, X_{M,n_0})$ of Eq. (1.5) have the distribution of μ_{n_0}, as defined in Eq. (7.3), Proposition 1.4 follows from the results of the previous section.

Proof (*Proof of* Proposition 1.4). The generating function $E(\boldsymbol{u})$ of Dyck paths satisfies the q-functional equation (2.6). Assumption 4.1 holds with $u_c = 1/4$ and $y_c = 1$. We have $f_{\mathbf{0},0} = -4$, $\gamma_{\mathbf{0}} = -1/2$, $\mu_i = (i+1)/4$ for $i = 1, \ldots, M-1$ and $\mu_0 = 1/8$. Thus, Proposition 4.5 yields Eq. (1.11). The distribution of the random variables $(X_{1,n_0}, \ldots, X_{M,n_0})$ of Eq. (1.5) is that of the probability measures μ_{n_0} in Eq. (7.3). By part (i) of Theorem 1.5, there exists a unique limit probability measure μ. We thus get Eq. (1.16) from Eq. (7.4). $\square$

The connection between Dyck paths and Brownian excursions in Proposition 1.1 leads, for a general q-functional equation, to an explicit characterisation of the limit probability measure μ, resp. of the associated limit random variable $(Y_1, \ldots, Y_M)$.

Proof (*Proof of claim (ii) in* Theorem 1.5.). For the given q-functional equation, let $F_0(\boldsymbol{\epsilon})$ be the generating function of the leading amplitudes $f_{\boldsymbol{k},0}$. For $k = 1, \ldots, M$ and $d_k \in \mathbb{R}$, define $G_0(\boldsymbol{\epsilon}) = F_0(\epsilon_1 d_1, \ldots, \epsilon_M d_M)$. An explicit calculation using Eq. (5.7) shows that the power series $G_0(\boldsymbol{\epsilon})$ satisfies the same type of differential equation as $F_0(\boldsymbol{\epsilon})$ does, with μ_0 replaced by $\mu_0 d_1$ and μ_i replaced by $\mu_i d_{i+1}/d_i$, where $i = 1, \ldots, M-1$. Now choose the values d_k, such that the equation for $G_0(\boldsymbol{\epsilon})$ is that of Dyck paths. Noting the relation between Dyck paths and Brownian excursions Eq. (1.7), we arrive at the values $c_k = 2^{(k+2)/2}/d_k$, for numbers c_k as in the claim of Theorem 1.5. $\square$

9. Concluding remarks

We finally stress three central aspects of our approach. Firstly, the approach yields a *universal* limit distribution — loosely spoken, it appears for all models, whose underlying functional equation has the same singularity structure. Such a result may be compared to a central limit theorem in probability theory. For example, models other than Dyck paths, which display a square root as dominant singularity of their size generating functions, are certain models of trees or polygons. For simply generated trees, q-functional equations appear when counting by number of vertices and by internal path length [50]. Using the above setup, moment recursions for the parameter "sum of kth powers of the vertex distances to the root" are obtained. For polygon models, q-functional equations appear when counting by perimeter and area [19,46], which is, for column-convex polygons, the sum of the column heights. The above setup gives moment recursions for the parameter "sums of kth powers of the column heights", in the limit of infinite (horizontal) perimeter.

Secondly, our approach is *algorithmic* — the moment recursion can finally be deduced from a straightforward calculation, by the method of dominant balance. Our approach also allows one to obtain corrections to the asymptotic behaviour, which cannot (easily) be deduced by other methods.

Thirdly, the approach is *flexible* — it may be applied to other classes to obtain a universal limit distribution, which depends only on the singularity structure of the functional equation. Our generating function approach is particularly suited for counting parameters, which decompose linearly under the cartesian product construction. Examples of such models with a rational generating function appear in [46]. Examples with an inverse square root appear in [39]. In particular, the discrete counterparts of Brownian motion, Brownian bridges, and Brownian meanders can be studied, see [39,48]. Also, universality questions for parameters related to left and right path lengths in trees [29,33,42,10] can be studied by our methods; compare with the discussion in [11].

Within the framework of simply generated trees, an alternative derivation of the above moment recursion could be obtained with the techniques of [28], where the different problem of the (generalised) Wiener index of trees was analysed. It would be interesting to consider how our methods can be adapted to that problem.

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